Exercise walk-through

5.5
Differential equations and numerical solutions

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

5.5 Differential equations and numerical solutions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS initial condition

Objective

Differential equations and numerical solutions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** : A specified value that selects a

particular solution from a family.

The method

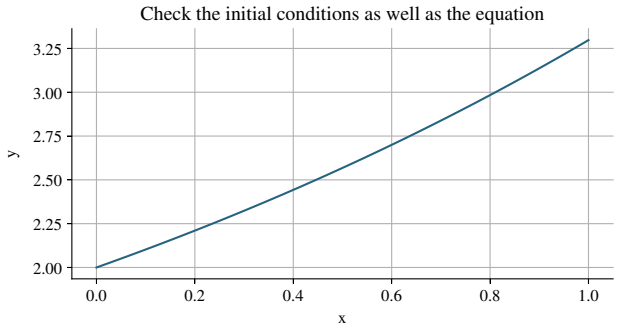
- For dy/dx=ky, separate variables: dy/y=k dx, giving y=Ae^(kx).
- Use an initial condition to find A.
- Euler's method takes y next=y+h f(x,y), with a chosen step h.

A worked case

$$\frac{dy}{dx} = ky, \qquad y = Ae^{kx}$$

Example:

With dy/dx=0.5y and y(0)=2, y=2e^(0.5x). Euler with h=0.2 gives y(0.2)≈2+0.2×1=2.2 and y(0.4)≈2.2+0.2×1.1=2.42. The exact second value is about 2.4428.



■ **Where it goes wrong**

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution y=0.
- Euler accuracy depends on step size and the equation.

Not true: Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

■ **Applying it**

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
- This is Further Pure scope; check the named unit.

■ **In your course**

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Initial condition**: A specified value that selects a particular solution from a family.
- For dy/dx=ky, separate variables: dy/y=k dx, giving y=Ae^(kx).
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

- **Write from a first Euler value of 4.4**
  - Use Euler with step 0.2 for y prime $= 0.5$ y, starting at y $= 4$.
  - Find the first two approximations. A different step size gives a different approximation.

Exercises 2.1·2.3

2.1 Using Euler with h=0.2 for y prime=0.5y,y_0=2, find y_1. Show the relation, substitution and units. [2]

2.2 For that approximation, find y_2. Show the relation, substitution and units. [2]

2.3 For y=2e^(0.5x), find y at x=0. Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Using Euler with h=0.2 for y prime=0.5y,y_0=2, find y_1. Show the relation, substitution and units. [2]

2.1 Euler uses $y_1=y_0+h\ f(y_0)=2+0.2\times0.5\times2=2.2$.

2.2 For that approximation, find y_2. Show the relation, substitution and units. [2]

2.2 Repeat from y_1: $y_2=2.2+0.2\times0.5\times2.2=2.42$.

2.3 For y=2e^(0.5x), find y at x=0. Show the relation, substitution and units. [2]

2.3 At x=0 the exponential is 1, so y=2.

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: Every step size gives exactly the same Euler approximation. [2]

2.5 For $dy/dx=0.5y$ with $y(0)=2$, what does the initial condition determine? Choose one answer.

- **A** The constant A in the family $y=Ae^{(0.5x)}$, giving $y=2e^{(0.5x)}$
- **B** The step size for Euler's method, which is set equal to $y(0)=2$
- **C** The growth constant k, which changes from 0.5 to 2 once $y(0)=2$ is used [1]

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: Every step size gives exactly the same Euler approximation. [2]

2.4 False. Euler's method depends on the step size; smaller steps usually give a more accurate value.

2.5 For $dy/dx=0.5y$ with $y(0)=2$, what does the initial condition determine? Choose one answer.

- **A** The constant A in the family $y=Ae^{(0.5x)}$, giving $y=2e^{(0.5x)}$
- **B** The step size for Euler's method, which is set equal to $y(0)=2$
- **C** The growth constant k, which changes from 0.5 to 2 once $y(0)=2$ is used [1]

2.5 A Separation gives the family $y=Ae^{(0.5x)}$ with a free constant. Substituting $x=0$, $y=2$ fixes $A=2$, selecting one particular solution from the family.

Exercise 3.1

3.1 Use Euler h=0.5 for y′=−y, y(0)=2 to estimate y(1). Compare with 2 e^−1 and state a limitation. [4]

Exercise 3.1

3.1 Use Euler h=0.5 for y′=−y, y(0)=2 to estimate y(1). Compare with 2 e^−1 and state a limitation. [4]

3.1 Updates 2→1→0.5; exact≈0.735759. Euler underestimates here; finite steps introduce error. Halve the step and compare convergence, rather than treating the numerical value as exact.