

Exercise walk-through

5.4

Substitution, parts and partial fractions

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

5.4 Substitution, parts and partial fractions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS integration by parts

Objective

Substitution, parts and partial fractions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

■ Can we integrate the two factors separately?

- A simple-looking product such as $x e^x$ cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** : An integration method based

on the derivative of a product.

■ The method

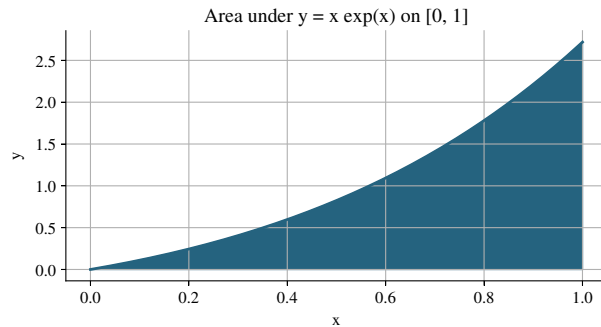
- Use substitution when an inner derivative appears as a factor.
 - By parts, $\int u v' = uv - \int u' v$.
 - For a rational function, divide first if needed, then decompose into partial fractions.
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■ A worked case

$$\int uv' dx = uv - \int u'v dx$$

Example:

For integral $x e^x dx$, take $u=x$ and $v' = e^x$. Then the integral is $x e^x - e^x + C$. For integral $2x/(x^2+1) dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1) + C$.



■ **Where it goes wrong**

- Choosing u and v prime well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: $\int x \cdot x \, dx = x^3/3$, not $(x^2/2)(x^2/2)$.

■ **Applying it**

- Check by differentiating.
 - For volumes of revolution around the x -axis use $V = \pi \int y^2 \, dx$; do not confuse the square of a function with the integral of the function.
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■ **In your course**

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Integration by parts:** An integration method based on the derivative of a product.
 - Use substitution when an inner derivative appears as a factor.
 - Watch for: choosing u and v prime well matters: the remaining integral should become simpler.
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■ Write from a volume factor of 18

- The line $y = 3$ is revolved about the x-axis from 0 to 2. Find the volume divided by π .
- The integral of a product is not the product of the integrals.

Exercises 2.1·2.3

2.1 Evaluate the integral of xe^x from 0 to 1. Show the relation, substitution and units. [2]

2.2 Find the coefficient of $\ln(x^2+1)$ in the integral of $2x/(x^2+1)$. Show the relation, substitution and units. [2]

2.3 Find the volume divided by π when $y=x$ is revolved about the x-axis from 0 to 3. Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Evaluate the integral of xe^x from 0 to 1. Show the relation, substitution and units. [2]

2.1 An antiderivative is $(x-1)e^x$. At 1 and 0 it gives 0 and -1, so the integral is 1.

2.2 Find the coefficient of $\ln(x^2+1)$ in the integral of $2x/(x^2+1)$. Show the relation, substitution and units. [2]

2.2 Substitution $w=x^2+1$ gives $dw=2x dx$, so the coefficient of $\ln(x^2+1)$ is 1.

2.3 Find the volume divided by π when $y=x$ is revolved about the x-axis from 0 to 3. Show the relation, substitution and units. [2]

2.3 $V/\pi = \int_0^3 x^2 dx = [x^3/3]_0^3 = 9$.

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: The integral of a product equals the product of its separate integrals. [2]

2.5 Why does the substitution $w=x^2+1$ fit the integral of $2x/(x^2+1)$? Choose one answer.

- **A** Because x^2+1 has no antiderivative, a substitution is the only option
- **B** Substitution works for every quotient, whatever the numerator is
- **C** The numerator $2x$ is exactly the derivative of the inner expression, so it cancels [1]

3.1 Evaluate integral of $x \ln x$ from 1 to e by parts and differentiate the antiderivative as a check. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: The integral of a product equals the product of its separate integrals. [2]

2.4 False. A product needs a method such as integration by parts or substitution: $\int x \cdot x \, dx = x^3/3$, not $(x^2/2)(x^2/2)$.

2.5 Why does the substitution $w=x^2+1$ fit the integral of $2x/(x^2+1)$? Choose one answer.

- **A** Because x^2+1 has no antiderivative, a substitution is the only option
- **B** Substitution works for every quotient, whatever the numerator is
- **C** The numerator $2x$ is exactly the derivative of the inner expression, so it cancels [1]

2.5 C With $w=x^2+1$, $dw=2x \, dx$, and the integral becomes $\int dw/w = \ln|w| + C$. Substitution is powerful precisely when the inner derivative appears as a factor.

3.1 Evaluate integral of $x \ln x$ from 1 to e by parts and differentiate the antiderivative as a check. [4]

3.1 Take $u=\ln x$, $dv=x \, dx$: antiderivative $(x^2/2)\ln x - x^2/4$. Bounds give $(e^2+1)/4$. Differentiation returns $x \ln x$; domain $x>0$ matters.