

Exercise walk-through

2.2

Exponentials, logarithms and modelling

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

2.2 Exponentials, logarithms and modelling

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS half-life

Objective

Exponentials, logarithms and modelling. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

■ Why does a decay model stay positive?

- A medicine concentration falls by the same percentage each hour. A constant subtraction would eventually predict a negative amount.
- This lesson studies **half-life** : The time for a decaying quantity to fall to half

its initial value.

■ The method

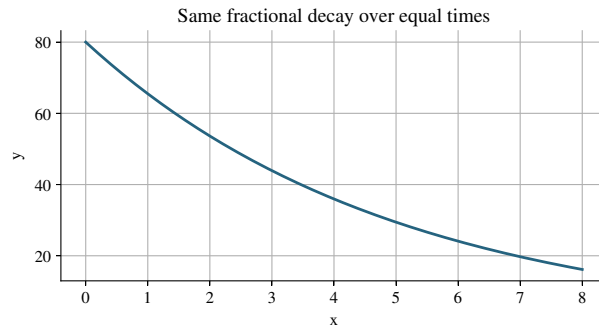
- For $y = Ae^{kt}$, k is a proportional rate.
 - Taking logs gives $\ln y = \ln A + kt$.
 - Logarithms require positive arguments, and $\log(x+y)$ is not $\log x + \log y$.
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■ A worked case

$$y = Ae^{kt}, \quad \ln y = \ln A + kt$$

Example:

If $y=80e^{(-0.2t)}$, $y=40$ gives $e^{(-0.2t)}=1/2$. Hence $t=\ln 2/0.2 \approx 3.466$. For $3^x=20$, $x=\ln 20/\ln 3$. Plotting $\ln y$ against t linearises this exponential model.



■ **Where it goes wrong**

- A fitted exponential is a model, not a guarantee.
- Specify the time units and range of use.
- A negative k describes decay; a negative starting amount usually contradicts the context.

Not true: For positive x and y , $\ln(x+y) \neq \ln x + \ln y$. The log of a sum has no simple law; the product rule is $\ln(xy) = \ln x + \ln y$.

■ **Applying it**

- Compare actual observations with the model.
- Systematic departures may indicate changing conditions.
- In a report, explain what the rate and initial value mean, rather than giving a bare equation.

■ **In your course**

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Half-life:** The time for a decaying quantity to fall to half its initial value.
 - For $y = Ae^{kt}$, k is a proportional rate.
 - Watch for: A fitted exponential is a model, not a guarantee.
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■ Write from 2 to the power 8

- Solve 2 to the power $x = 256$.
- The sum of $\ln 2$ and $\ln 9$ is $\ln 18$, not $\ln$ of the sum.

Exercises 2.1·2.3

2.1 Solve $2^x = 32$. Show the relation, substitution and units. [2]

2.2 Find $\ln(e^3)$. Show the relation, substitution and units. [2]

2.3 Find the half-life of $80e^{(-0.2t)}$, to three decimal places. Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Solve $2^x = 32$. Show the relation, substitution and units. [2]

2.1 $32 = 2^5$, so $x = 5$.

2.2 Find $\ln(e^3)$. Show the relation, substitution and units. [2]

2.2 The natural logarithm undoes exponentiation: $\ln(e^3) = 3$.

2.3 Find the half-life of $80e^{(-0.2t)}$, to three decimal places. Show the relation, substitution and units. [2]

2.3 Set $e^{(-0.2t)} = 1/2$. Then $t = \ln(2)/0.2 \approx 3.466$ to three decimal places.

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: For positive x and y , $\ln(x+y)=\ln x+\ln y$. [2]

2.5 Why is $\ln(x+y)=\ln x+\ln y$ wrong? Choose one answer.

- **A** Logarithms apply only to integers, so $\ln(x+y)$ is undefined for most x and y
- **B** The law is $\ln(x+y)=\ln x \times \ln y$, so adding the logs gives the wrong value
- **C** The log of a SUM has no such law; the law covers products: $\ln(xy)=\ln x+\ln y$ [1]

3.1 A fictional quantity follows $80 \exp(-0.2 t)$. Find when it reaches 20, retaining an exact expression; explain why the half-life does not depend on 80. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: For positive x and y , $\ln(x+y)=\ln x+\ln y$. [2]

2.4 False. The log of a sum has no simple law; the product rule is $\ln(xy)=\ln x+\ln y$.

2.5 Why is $\ln(x+y)=\ln x+\ln y$ wrong? Choose one answer.

- **A** Logarithms apply only to integers, so $\ln(x+y)$ is undefined for most x and y
- **B** The law is $\ln(x+y)=\ln x \times \ln y$, so adding the logs gives the wrong value
- **C** The log of a SUM has no such law; the law covers products: $\ln(xy)=\ln x+\ln y$ [1]

2.5 C Logarithms convert multiplication into addition: $\ln(xy)=\ln x+\ln y$. Try $x=y=1$: $\ln 2 \approx 0.69$ but $\ln 1+\ln 1=0$, so the claimed law fails.

3.1 A fictional quantity follows $80 \exp(-0.2 t)$. Find when it reaches 20, retaining an exact expression; explain why the half-life does not depend on 80. [4]

3.1 $t=\ln 4/0.2 \approx 6.931$ time units. Half-life $\ln 2/0.2$ follows from a ratio $1/2$; the initial multiplier cancels. Extrapolation assumes the exponential model continues.