

Exercise walk-through

1.7

Series, finite differences and Taylor expansions

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

1.7 Series, finite differences and Taylor expansions

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS Taylor polynomial

Objective

Series, finite differences and Taylor expansions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

■ Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
- This lesson studies **Taylor polynomial** : A polynomial formed from a

function's derivatives at a chosen centre.

■ The method

- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
 - At $a=0$ it is a Maclaurin expansion.
 - Finite sums may also simplify by cancellation or standard sum formulae.
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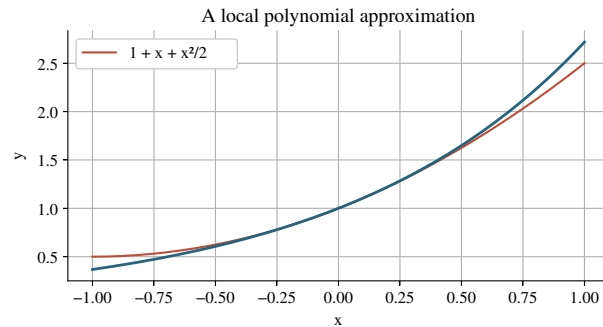
■ A worked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

∠:

Example:

The first three terms of e^x are $1+x+x^2/2$. At $x=0.1$ this gives 1.105, close to $e^{0.1} \approx 1.1051$.
For the sum of k^2 from $k=1$ to 5, $n(n+1)(2n+1)/6$ gives $5 \times 6 \times 11 / 6 = 55$.



■ **Where it goes wrong**

- The factorial belongs in the denominator of every Taylor coefficient.
- The expansion centre is not always zero.
- An approximation close to its centre can be poor far away.

Not true: A Taylor polynomial is always exact for the original function at every input. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

■ **Applying it**

- For a telescoping series, write several terms and identify both surviving ends.
- For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.

■ **In your course**

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Taylor polynomial:** A polynomial formed from a function's derivatives at a chosen centre.
 - About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
 - Watch for: the factorial belongs in the denominator of every Taylor coefficient.
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■ Write from a square sum of 30

- Find the sum of k squared from $k = 1$ to 4.
- Use $1 + x + x^2/2$ to estimate e to the power 0.2. A Taylor polynomial is not exact at every input.

Exercises 2.1·2.3

2.1 Use $1+x+x^2/2$ to estimate $e^{0.1}$. Show the relation, substitution and units. [2]

2.2 Find the sum of k^2 for $k=1$ to 5. Show the relation, substitution and units. [2]

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x . Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Use $1+x+x^2/2$ to estimate $e^{0.1}$. Show the relation, substitution and units. [2]

2.1 The truncated polynomial is $1+0.1+0.1^2/2=1.105$.

2.2 Find the sum of k^2 for $k=1$ to 5 . Show the relation, substitution and units. [2]

2.2 Add the squared terms: $1+4+9+16+25=55$.

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x . Show the relation, substitution and units. [2]

2.3 The second derivative of e^x at zero is 1 ; divide by $2!=2$ to get $1/2$.

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.5 Why does $1+x+x^2/2$ only approximate e^x ? Choose one answer.

- **A** The factorials belong in the denominators, so the true series grows faster
- **B** The Maclaurin series continues with $x^3/6$ and beyond; truncating drops those terms
- **C** e^x is not a function, so no polynomial can represent it exactly [1]

3.1 Approximate $e^{0.2}$ with $1+x+x^2/2$. Give a remainder bound using $e^\xi x^3/6$ with $0 < \xi < 0.2$ and explain what the bound certifies. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.4 False. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

2.5 Why does $1+x+x^2/2$ only approximate e^x ? Choose one answer.

- **A** The factorials belong in the denominators, so the true series grows faster
- **B** The Maclaurin series continues with $x^3/6$ and beyond; truncating drops those terms
- **C** e^x is not a function, so no polynomial can represent it exactly [1]

2.5 B The full series is infinite: $e^x=1+x+x^2/2!+x^3/3!+....$ Stopping after three terms discards the tail, so the result is an approximation good only near the centre $x=0$.

3.1 Approximate $e^{0.2}$ with $1+x+x^2/2$. Give a remainder bound using $e^\xi x^3/6$ with $0<\xi<0.2$ and explain what the bound certifies. [4]

3.1 Approximation 1.22; $0<R<e^{0.2}(0.2)^3/6\approx 0.001629$. The quadratic underestimates and the exact value lies between 1.22 and 1.221629. The bound depends on the stated Taylor theorem conditions.