

Exercise walk-through

1.6

Binomial expansion and valid approximations

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

1.6 Binomial expansion and valid approximations

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS binomial coefficient

Objective

Binomial expansion and valid approximations. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

■ How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
- This lesson studies **binomial coefficient** : The number of ways to choose

a specified number of objects from a set.

■ The method

- For positive integer n , $(a+b)^n$ is a finite binomial expansion.
 - For noninteger n , expand $(1+x)^n$ as $1+nx+n(n-1)x^2/2+\dots$ with $|x|<1$.
 - Factor out constants before using this form.
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■ A worked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

Example:

$(1+2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$. For $(1+x)^{-1}$, the first three terms are $1 - x + x^2$; at $x=0.1$ this gives 0.91 versus the exact $1/1.1 \approx 0.909091$.

$$(1 + 2x)^5$$

Coefficients: 1, 10, 40, 80, 80, 32

The x^2 coefficient includes $(2x)^2$

■ **Where it goes wrong**

- An expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$.
- Positive-integer expansions are finite and do not have that convergence restriction.

Not true: Every binomial expansion is a finite polynomial, including powers that are not positive integers. For a negative or fractional power the expansion is an infinite series, valid only for $|x| < 1$.

■ **Applying it**

- State the range of validity alongside an approximation.
- Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x .

■ **In your course**

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Binomial coefficient:** The number of ways to choose a specified number of objects from a set.
 - For positive integer n , $(a+b)^n$ is a finite binomial expansion.
 - Watch for: an expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$.
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■ Write from a coefficient of 54

- Find the coefficient of x squared in $(1 + 3x)$ to the power 4.
- A binomial expansion is finite only for a non-negative integer power.

Exercises 2.1·2.3

2.1 Find the coefficient of x^2 in $(1+2x)^5$. Show the relation, substitution and units. [2]

2.2 Use $1-x+x^2$ to estimate $1/(1+x)$ at $x=0.1$. Show the relation, substitution and units. [2]

2.3 How many terms are in the full expansion of $(a+b)^5$? Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Find the coefficient of x^2 in $(1+2x)^5$. Show the relation, substitution and units. [2]

2.1 Choose two of five factors and include 2^2 : $C(5,2) \times 4 = 40$.

2.2 Use $1-x+x^2$ to estimate $1/(1+x)$ at $x=0.1$. Show the relation, substitution and units. [2]

2.2 Substitute 0.1: $1-0.1+0.01=0.91$.

2.3 How many terms are in the full expansion of $(a+b)^5$? Show the relation, substitution and units. [2]

2.3 The powers run from 0 through 5, giving six terms.

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: Binomial coefficient means the number of ways to choose a specified number of objects from a set. [2]

2.5 For the infinite expansion of $(1+2x)^{-1}$, which condition is required? Choose one answer.

- **A** No condition: it always converges
- **B** Only $|x| < 1$
- **C** $|2x| < 1$, so $|x| < 1/2$

[1]

3.1 Use $1-x+x^2$ to approximate $1/(1+x)$ at $x=0.2$. Find the exact error algebraically and explain its sign. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: Binomial coefficient means the number of ways to choose a specified number of objects from a set. [2]

2.4 True. True: The number of ways to choose a specified number of objects from a set.

2.5 For the infinite expansion of $(1+2x)^{-1}$, which condition is required? Choose one answer.

- **A** No condition: it always converges
- **B** Only $|x| < 1$
- **C** $|2x| < 1$, so $|x| < 1/2$ [1]

2.5 C The validity condition $|u| < 1$ applies to the whole bracket term $u=2x$, so $|x| < 1/2$. Using $|x| < 1$ would wrongly accept $x=0.9$, where the series diverges.

3.1 Use $1-x+x^2$ to approximate $1/(1+x)$ at $x=0.2$. Find the exact error algebraically and explain its sign. [4]

3.1 Approximation 0.84; exact $5/6$. Identity $(1-x+x^2)-1/(1+x)=x^3/(1+x)$, so error $1/150 \approx 0.006667$, positive. The infinite geometric expansion requires $|x| < 1$.