Coordinate operators, separated modes and stable updates

GRE undergraduate preparation · local supported objectives

Coordinate operators, separation of variables and stable steps

- Use the **Laplacian 拉普拉斯算子** and divergence in spherical and cylindrical coordinates.
- Solve wave and heat equations by **separation of variables 分离变量法**.
- Find the **stability 稳定性** limit of the explicit Euler method.
- Check that a numerical answer is accurate, not only stable.

Radial operators

- $1/r$ has zero **Laplacian** for $r > 0$: the potential of a point charge.
- $\hat{r}/r^2$ has zero divergence for $r > 0$: the field of a point charge.
- r^2 has Laplacian $2 + 4 = 6$.
- In cylindrical coordinates the radial Laplacian is $\frac{1}{s}\frac{d}{ds}\left(s\frac{df}{ds}\right)$, so $\ln s$ is the harmonic function.

$$\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = f'' + \frac{2}{r}f', \qquad \nabla \cdot (A_r\hat{r}) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 A_r\right)$$

Separation of variables

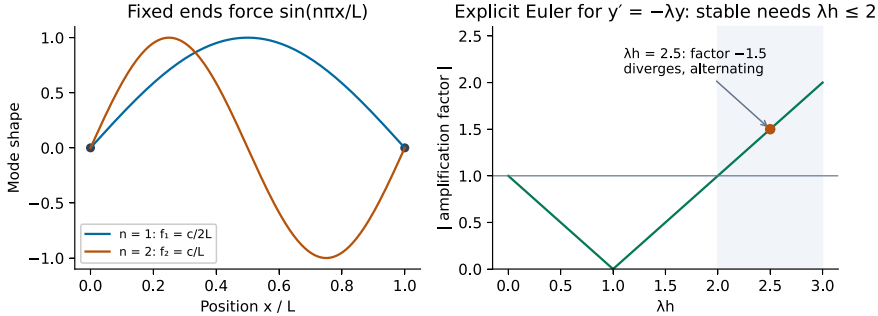
- $X(0) = X(L) = 0$ forces $X = \sin(n\pi x/L)$, so the frequencies are $f_n = nc/2L$.
- The general solution is a sum of these modes, with amplitudes set by the initial shape.
- For the heat equation $u_t = Du_{xx}$, each mode decays as $e^{-D(n\pi/L)^2 t}$: high modes die out fastest.

Explicit Euler stability

- The method is stable when $|1 - \lambda h| \le 1$, that is $0 < h \le 2/\lambda$.
- For $1/\lambda < h < 2/\lambda$ the result oscillates in sign but still decays; above $2/\lambda$ it grows.

$$y_{n+1} = (1 - \lambda h)\, y_n$$

Figure: Explicit Euler stability



- Identify the axes, units, directions and the stated model.

Stable is not accurate

- Explicit Euler has an error roughly proportional to h. Halve the step and the error should roughly halve.
- Check that λh is dimensionless, compare two step sizes, and test a case with a known answer.

Worked example: operators and modes

- For $f = 1/r$: $f'' = 2/r^3$ and $\frac{2}{r} f' = -2/r^3$, so $\nabla^2 f = 0$. Using f'' alone gives the wrong $2/r^3$.
- A string with $L = 0.50$ m and $c = 100$ m/s: $f_1 = 100/(2 \times 0.50) = 100$ Hz, and $f_3 = 300$ Hz.
- $\lambda = 10\ \mathrm{s}^{-1}$ with $h = 0.25$ s: the factor is $1 - 2.5 = -1.5$, so the solution grows with alternating sign.

Worked example: refining a step

- $h = 0.2$: factor 0.2, five steps, $8 \times 0.2^5 = 0.0026$.
- $h = 0.1$: $8 \times 0.6^{10} = 0.048$. $h = 0.05$: $8 \times 0.8^{20} = 0.092$.
- All three are stable ($h < 0.5$), but only small steps approach the right answer.

Common errors

- Writing $\nabla^2 f = f''$ in spherical or cylindrical coordinates.
- Summing modes that do not satisfy the boundary conditions.
- Treating $h < 2/\lambda$ as a guarantee of accuracy.

Common errors (2)

- Mixing units, so that λh is not dimensionless.
- Forgetting that high modes of the heat equation decay fastest.

Key terms

- **Laplacian**: ∇^2, the sum of second derivatives, written in the form that belongs to the coordinate system.
- **separation of variables**: Solving a partial differential equation by writing the unknown as a product of one-variable factors.
- **stability**: A numerical method's property of not amplifying errors from step to step.

Check yourself

- What is the **Laplacian** of r^3 in spherical coordinates?
- What is the largest stable explicit Euler step for $y' = -50y$?
- A string fixed at both ends has $f_1 = 220$ Hz. What is f_2?