

Coordinate operators, separated modes and stable updates

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

Writing the Laplacian of a radial function as f'' drops a term; the operator belongs to the coordinates, not the symbol. Use the divergence and Laplacian in the coordinate system selected by the symmetry Solve a simple separated boundary-value mode problem Check convergence, units and stability of an original numerical update

Match operator to symmetry

Reasoning

Choose coordinates by symmetry before writing operators. For a radial function in spherical coordinates, $\nabla^2 f = f'' + (2/r)f'$, and the divergence of a radial field $A = A_r \hat{r}$ is $(1/r^2)d(r^2 A_r)/dr$. Check: $A = k \hat{r}/r^2$ has divergence $(1/r^2)dk/dr = 0$ away from the origin, and $f = 1/r$ satisfies $\nabla^2 f = f'' + 2f'/r = 2/r^3 - 2/r^3 = 0$ away from the origin. Cartesian forms apply only to Cartesian dependences.

Separate the modes

Reasoning

Separation of variables turns a boundary-value problem into ordinary modes. For a string fixed at both ends, write $u=X(x)T(t)$; the wave equation gives $X''/X=(1/c^2)T''/T=-k^2$, the fixed ends force $X=\sin(n\pi x/L)$, and the allowed frequencies are $\omega_n=n\pi c/L$, $f_n=nc/(2L)$. Each mode must satisfy the boundary conditions before any sum over modes.

Bound the step

Reasoning

An explicit Euler update for $y' = -\lambda y$ is $y_{n+1} = (1 - \lambda h)y_n$, bounded when $|1 - \lambda h| \leq 1$, i.e. $0 < h \leq 2/\lambda$ for $\lambda > 0$; asymptotic decay needs $0 < h < 2/\lambda$. With $\lambda = 10$ /s, h must not exceed 0.2 s; $h = 0.25$ s multiplies by -1.5 each step and diverges with alternating sign. Stability is a property of the recurrence, not of the true solution.

Verify the answer

Reasoning

Check a numerical answer three ways before trusting it: units of every term (λh must be dimensionless), convergence under refinement (halve h and compare; explicit Euler error is first order), and a limiting case with a known exact answer. For $y' = -4y$ with $y_0 = 8$ and $h = 0.2$, five steps give $8 \times 0.2^5 = 0.00256$ versus exact $8e^{-4} \approx 0.1465$: stable but inaccurate, so refine.

Worked reasoning 1

Reasoning

A spherical radial field has divergence

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

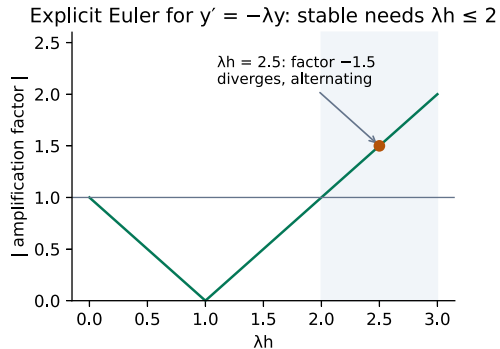
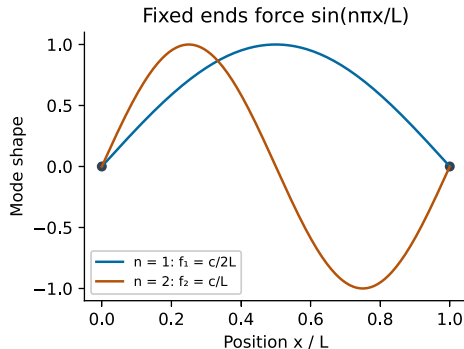
$$\nabla \cdot (A_r \hat{r}) = r^{-2} \frac{d}{dr} (r^2 A_r).$$

Worked reasoning 3

Reasoning

For $A_r = Cr$, it gives $3C$, not C . Fixed string ends require $X(0) = X(L) = 0$, hence $X_n = \sin(n\pi x/L)$ with positive integer n . Euler for $\dot{y} = -\lambda y$ gives $y_{n+1} = (1 - \lambda h)y_n$. For positive h , bounded absolute stability permits $h \leq 2/\lambda$; decay to zero requires the strict inequality $0 < h < 2/\lambda$. At the upper endpoint the numerical solution alternates without decay. Stability does not establish accuracy.

Read the original diagram



- Original GRE teaching diagram: Coordinate operators, separated modes and stable updates
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

Apply explicit and implicit Euler to $\dot{y} = -\lambda y$, λ positive. Derive both amplification factors. Find the step range for nonnegative monotone explicit decay, and decide whether implicit Euler is exact for arbitrarily large steps. **[5]**

Independent transfer · reasoning

Reasoning

Explicit factor is $1 - \lambda h$. Nonnegative monotone decay needs $0 < h \leq 1/\lambda$; at equality one step reaches zero, unlike the exact exponential. Implicit Euler solves $y_{n+1} = y_n - \lambda h y_{n+1}$, so its factor is $1/(1 + \lambda h)$, between zero and one for every positive h . It is stable but only first-order accurate; refine h at fixed final time and compare with $e^{-\lambda t}$. Award factors, monotonicity range, endpoint, implicit stability and accuracy/refinement distinction.

Check assumptions and limits

Reasoning

Applying the Cartesian Laplacian to a radial function, summing modes that violate the boundary conditions, or calling a bounded-looking run converged without refining the step. Match operator, boundaries and stability condition to the actual problem.

Keep the technical distinctions

Reasoning

separation of variables : Solving a partial differential equation by writing the unknown as a product of single-variable factors. stability condition : The step-size restriction that keeps a numerical recurrence from amplifying errors.

Check yourself

- Define separation of variables and give its condition of use.
- Define stability condition and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.