Weak perturbations and degenerate subspaces

GRE undergraduate preparation · local supported objectives

Weak perturbations and degenerate subspaces

- Find the **first-order shift** 一级修正 as an **expectation value** 期望值.
- Use symmetry to show a first-order shift is zero.
- Know the sign and size of the **second-order shift** 二级修正.
- Diagonalise the perturbation in a **degenerate** 简并的 subspace.

First order

- The shift is W averaged over the unperturbed probability density.
- Infinite well $0 < x < L$: every state has $\langle x \rangle = L/2$, so $W = \gamma x$ shifts every level by $\gamma L/2$.

$$E_n^{(1)} = \lambda \langle n | W | n \rangle = \lambda \int |\psi_n|^2 \, W \, dx$$

Symmetry

- In a symmetric well, a tilt $\gamma(x - L/2)$ gives zero first-order shift for every state, though the wavefunctions do change.
- An even perturbation such as $\eta(x - L/2)^2$ gives a positive shift that depends on n.

Second order

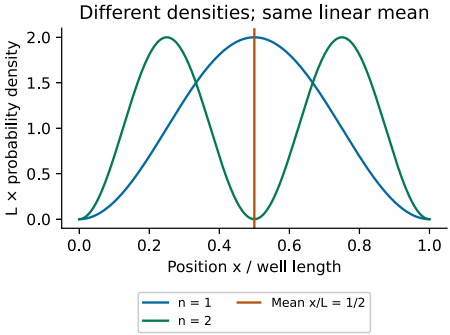
- For the ground state every denominator is negative, so $E^{(2)} \leq 0$: the ground state is always pushed down.
- The method works only when each coupling $|\lambda W_{mn}|$ is small compared with the gap $|E_n - E_m|$; nearly degenerate levels break it.

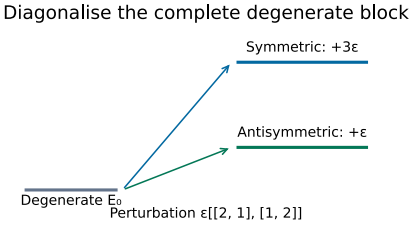
$$E_n^{(2)} = \sum_{m \neq n} \frac{|\lambda W_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

Degenerate levels

- Reading only the diagonal elements gives the wrong answer.
- A perturbation that is a multiple of the identity in the subspace shifts all the states equally and never splits the level.
- This is how the Stark effect splits the degenerate $n = 2$ level of hydrogen.

Figure: Degenerate levels





- Diagonalise the whole degenerate block: the eigenvalues give the first-order shifts
- Read the axes and units first.

Worked example: a tilted well

- $\langle x \rangle = 1.0$ nm for every state, so every level rises by $0.050 \times 1.0 = 0.050$ eV.
- Measure the tilt from the centre instead, $\gamma(x - L/2)$: the first-order shift is zero for every state.
- The two answers differ only by the constant $\gamma L/2$, which shifts all levels equally.

Worked example: a degenerate pair

- Eigenvalues: 3ε, for $(1,1)/\sqrt{2}$, and ε, for $(1,-1)/\sqrt{2}$.
- Reading the diagonal would give 2ε twice, with no splitting.
- A harmonic oscillator with $W = \lambda x$ has zero first-order shift (odd perturbation); the exact shift $-\lambda^2/(2m\omega^2)$ is second order and negative, as the rule says.

Common errors

- Using the diagonal elements alone in a degenerate subspace.
- Forgetting that odd perturbations give zero first-order shifts in symmetric states.
- Giving the ground state a positive **second-order shift**.

Common errors (2)

- Applying perturbation theory when the coupling is as large as the level gap.
- Thinking a constant perturbation can split a degenerate level.

Key terms

- **perturbation**: A small extra term added to a Hamiltonian that has already been solved.
- **first-order shift**: The energy change $\langle n|W|n\rangle$, the perturbation averaged over the unperturbed state.
- **expectation value**: The average result of measuring a quantity over many identical preparations.
- **second-order shift**: The energy change from couplings to other states, divided by the energy gaps.

Check yourself

- A perturbation is a constant c. What is the first-order shift of every level?
- A degenerate pair feels εI, a multiple of the identity. Does the level split?
- What is the sign of the **second-order shift** of a ground state?