

Weak perturbations and degenerate subspaces

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A zero first-order energy shift can coexist with a changed state and a nonzero higher-order shift. Calculate nondegenerate first-order energy shifts as normalised expectation values Use symmetry and matrix elements to assess weak-coupling limits Diagonalise the perturbation within an exactly degenerate subspace

Weight the perturbation by probability

Reasoning

Let $H=H_0+\lambda W$, where λ is a small dimensionless parameter and the eigenstates of H_0 are known and normalised. For a nondegenerate level n , the first-order energy correction is $\lambda\langle n|W|n\rangle$. In position space this is $\lambda\int\psi_n^*(x)W(x)\psi_n(x)dx$ for a multiplicative potential, with integration over the allowed domain. A potential value at one point is not an expectation value; the state's probability density weights the whole domain. For an infinite well $0<x<L$, $\psi_n=\sqrt{2/L}\sin(n\pi x/L)$. Reflection about its centre makes $\langle x\rangle=L/2$, so a weak added potential γx has shift $\gamma L/2$ for every nondegenerate well level at first order. γ has units of energy per length.

Use parity carefully

Reasoning

For a perturbation odd about the centre, such as $\gamma(x-L/2)$, the probability density of an unperturbed well eigenstate is even, so its diagonal expectation vanishes. A centred quadratic perturbation $\eta(x-L/2)^2$ instead gives shift $\eta L^2[1/12 - 1/(2\pi^2 n^2)]$, which is positive for $\eta > 0$ and depends on n . Parity makes the first-order integral zero only for the specified state and operator symmetry. Off-diagonal matrix elements can still change the wavefunction. Do not turn a symmetry cancellation into a claim that every energy correction vanishes or that the original state remains exact.

Compare coupling with level gaps

Reasoning

For a nondegenerate state, the leading admixture of another unperturbed state m is proportional to $\lambda W_{mn}/(E_n^0 - E_m^0)$. The useful smallness condition therefore compares coupling matrix elements with the relevant level separations, not just with an arbitrary absolute energy zero. To second order, the energy correction contains $\sum_{(m \neq n)} |\lambda W_{mn}|^2 / (E_n^0 - E_m^0)$. For the lowest nondegenerate state, all these denominators are negative, so the second-order correction is nonpositive in this model. Near a degeneracy, a small denominator defeats the nondegenerate expansion; use a coupled subspace instead of dividing by zero.

Resolve a degenerate subspace

Reasoning

If a level of H_0 is exactly degenerate, choose an orthonormal basis within that subspace and form the Hermitian matrix of the perturbation there. Its eigenvalues are the first-order energy shifts, and its eigenvectors specify the combinations that diagonalise the leading splitting. For a two-state subspace with perturbation $\varepsilon[[2,1],[1,2]]$, the normalised symmetric and antisymmetric combinations have shifts 3ε and ε . Reading only the two diagonal entries would incorrectly predict two shifts of 2ε . A common scalar multiple of the identity shifts both states equally and does not split their degeneracy; off-subspace couplings can matter at higher order.

Worked reasoning 1

Reasoning

A degenerate subspace requires matrix diagonalisation before first-order shifts. Suppose the perturbation restricted to two states is $\epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

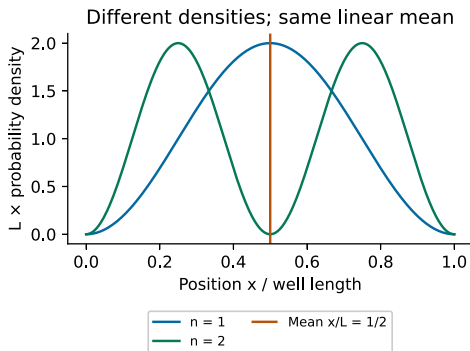
$$\det(W - wI) = (2\epsilon - w)^2 - \epsilon^2 = 0.$$

Worked reasoning 3

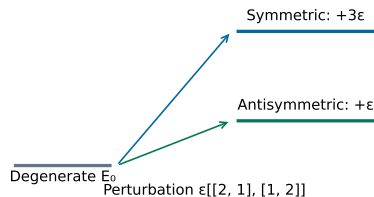
Reasoning

Shifts are epsilon and 3 epsilon. Normalised states are $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$. Reading only the diagonal entries would miss the splitting. Couplings to other levels must remain weak relative to their energy gaps.

Read the original diagram



Diagonalise the complete degenerate block



- Original GRE teaching diagram: Weak perturbations and degenerate subspaces
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

A two-state Hamiltonian is $\begin{pmatrix} 0 & \epsilon \\ \epsilon & \Delta \end{pmatrix}$, with Δ positive and $|\epsilon| \ll \Delta$. Find the exact eigenvalues and leading ground-energy shift. Explain why a zero diagonal perturbation is insufficient to predict zero shift. **[5]**

Independent transfer · reasoning

Reasoning

Characteristic equation is $E^2 - \Delta E - \epsilon^2 = 0$. $E_{\pm} = (\Delta \pm \sqrt{\Delta^2 + 4\epsilon^2})/2$. Expand the lower root: $E_- \approx -\epsilon^2/\Delta$. The first-order shift is zero, but off-diagonal mixing produces a negative second-order shift. Award equation, roots, expansion, sign and mixing/validity. At $\Delta = 0$ use degenerate theory instead.

Check assumptions and limits

Reasoning

Use normalised state weights, keep perturbation units, and diagonalise a degenerate block. First-order cancellation and exact invariance are different claims.

Keep the technical distinctions

Reasoning

first-order energy shift : Leading weak-perturbation correction given by the unperturbed state's expectation value. degenerate perturbation theory : Method that first diagonalises the perturbation within an unperturbed degenerate subspace.

Check yourself

- Define first-order energy shift and give its condition of use.
- Define degenerate perturbation theory and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.