

Quantum occupations and limits of equipartition

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A fermionic energy level can contain several particles when it contains several distinct states; the exclusion rule applies per complete state. Compute mean occupation per complete quantum state for fermions and bosons Count allowed identical-particle occupations and identify the classical dilute limit Compare classical quadratic-mode heat capacity with a quantum oscillator response

Assign occupation to complete states

Reasoning

For noninteracting particles in thermal and particle exchange equilibrium, use chemical potential μ and $x = (\varepsilon - \mu)/(kBT)$. Mean occupation of one complete state is $n_F = 1/(\hat{e}^x + 1)$ for fermions and $n_B = 1/(\hat{e}^x - 1)$ for bosons. Fermionic occupation of that state is zero or one; its mean lies between them. Bosonic occupation can exceed one. A level of degeneracy g has total mean g times the single-state occupation when its states share the same energy. Count spin as part of a complete state. The Bose denominator requires $\varepsilon > \mu$ for the ordinary finite expression, with the ground-state limit treated separately. For equilibrium photons $\mu = 0$ because photon number is not conserved; do not set $\mu = 0$ for every material particle gas.

Count indistinguishable configurations

Reasoning

Count occupation patterns rather than labelling identical particles. Two identical fermions distributed among four distinct complete states have $\text{choose}(4,2)=6$ allowed patterns. Two identical bosons among those states have $\text{choose}(4+2-1,2)=10$ patterns because both may share one state. Two labelled distinguishable particles would have $4^2=16$ assignments. These are different counting models, not three interchangeable answers to the same specification. If a question supplies spin degeneracy, first decide whether its stated number counts complete states or just orbital levels. A Pauli prohibition on two identical complete states does not prohibit opposite-spin fermions in one spatial orbital.

Check the dilute approximation

Reasoning

When x is large and positive, occupation is small and both denominators are dominated by e^x : $n_F \approx n_B \approx e^{-x}$, the Maxwell-Boltzmann dilute limit. At $x = \ln 4$ the means are $1/5$ for fermions, $1/3$ for bosons and $1/4$ in the classical approximation; the difference is still significant. A large total particle number alone does not justify classical statistics: density, temperature and accessible states control occupation. For fermions at low T , states below μ become nearly occupied and those above nearly empty. At $\varepsilon = \mu$ a fermionic state has mean one half; inserting that value into the ordinary Bose formula would instead produce a divergent denominator and requires different limiting treatment.

Test quantum versus classical modes

Reasoning

Classical equipartition assigns $kBT/2$ of mean energy to each independent quadratic term in an equilibrated Hamiltonian. A monatomic ideal gas has three translational terms, giving $U=3NkBT/2$ and $C_V=3NkB/2$. One classical one-dimensional harmonic oscillator has kinetic and potential terms, giving mean kBT and $C=kB$. For a quantum oscillator with fixed spacing $\varepsilon=\hbar\omega$, the thermal energy above its temperature-independent zero point is $\varepsilon/(\hat{e}^x-1)$, now $x=\varepsilon/(kBT)$, and $C/kB=x^2\hat{e}^x/(\hat{e}^x-1)^2$. At high T it approaches the classical value; at low T excitation freezes out and $C\rightarrow 0$. A zero-point energy $\varepsilon/2$ shifts U but not C . Molecular rotational/vibrational contributions likewise need their energy scales checked before assigning classical quadratic terms.

Worked reasoning 1

Reasoning

Occupation number refers to one complete single-particle state. For $x = (\epsilon - \mu)/(k_B T) = \ln 4$,

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

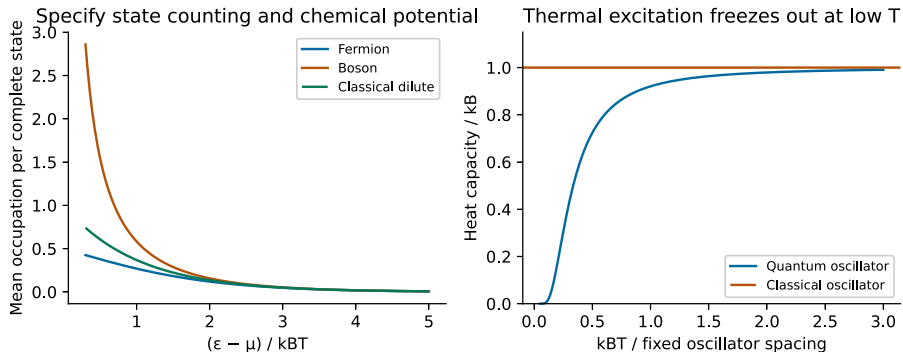
$$\bar{n}_F = (e^x + 1)^{-1} = 1/5, \quad \bar{n}_B = (e^x - 1)^{-1} = 1/3.$$

Worked reasoning 3

Reasoning

The classical dilute approximation gives $e^{-x} = 1/4$; it is not exact here. A g -fold level has mean total occupation $g\bar{n}$, so a fermionic level can contain more than one particle while each complete state still obeys exclusion.

Read the original diagram



- Original GRE teaching diagram: Quantum occupations and limits of equipartition
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

Three identical particles access four complete states with no energy restriction. Count occupation patterns for fermions and bosons, and compare with three labelled particles. Explain each counting rule. **[4]**

Independent transfer · reasoning

Reasoning

Fermions select three of four states: $\binom{4}{3} = 4$. Bosons distribute three indistinguishable particles among four states: $\binom{3+4-1}{3} = 20$ by stars and bars. Labelled particles each choose any state: $4^3 = 64$. Award each count and exclusion/indistinguishability reasoning. Spin is already included in complete states.

Check assumptions and limits

Reasoning

Exclude two fermions from one complete state, not from an entire degenerate energy level. State which x is used; chemical-potential occupations and fixed oscillator excitation formulas are different models.

Keep the technical distinctions

Reasoning

mean occupation : Ensemble average number of particles in one complete quantum state or a specified group of states. equipartition : Classical equilibrium rule assigning $kBT/2$ to each independent quadratic Hamiltonian term.

Check yourself

- Define mean occupation and give its condition of use.
- Define equipartition and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.