Constraints, variational equations and cyclic coordinates

GRE undergraduate preparation · local supported objectives

Constraints, variational equations and cyclic coordinates

- Choose **generalised coordinates 广义坐标** that respect the constraints.
- Write and differentiate the **Euler–Lagrange equation 欧拉 – 拉格朗日方程** correctly.
- Spot a **cyclic coordinate 循环坐标** and its conserved momentum.
- Find small-oscillation frequencies and equilibrium points.

Coordinates that respect constraints

- Then $v^2 = l^2\dot{\theta}^2$, $T = \frac{1}{2}ml^2\dot{\theta}^2$ and $V = mgl(1 - \cos\theta)$.
- A **holonomic 完整的** constraint is a relation among the coordinates; each one removes a coordinate.
- Constraints on velocities, such as rolling on a surface, may not reduce to coordinate relations and need more care.

The Euler–Lagrange equation

- Pendulum: $ml^2\ddot{\theta} + mgl\sin\theta = 0$, with no tension in sight.
- Differentiate every factor that changes with time: $\frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

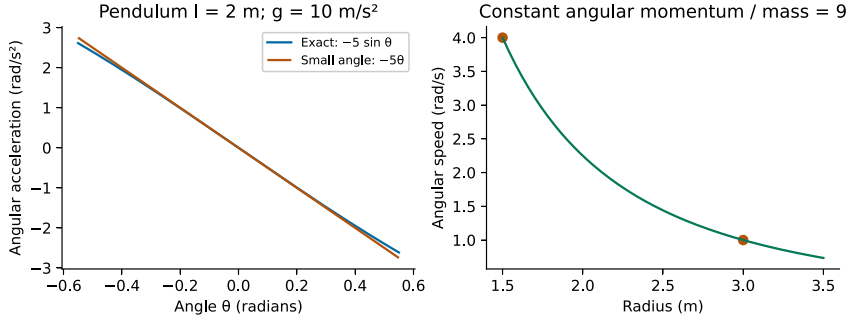
Cyclic coordinates

- Central force: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ contains no θ, so $p_\theta = mr^2\dot{\theta}$ is constant.
- Halving r multiplies $\dot{\theta}$ by 4.
- The radial equation is $m\ddot{r} = mr\dot{\theta}^2 - V'(r)$: the first term is the centrifugal effect of the rotation.

Small oscillations

- Pendulum: $V'' = mgl$ and $M = ml^2$, so $\omega = \sqrt{g/l}$.
- If $V''(q_0) < 0$, the equilibrium is unstable: report the instability, not an imaginary frequency.

Figure: Small oscillations



- Identify the axes, units, directions and the stated model.

Worked example: a 2 m pendulum

- At $\theta = 0.10$ rad: $\ddot{\theta} = -0.499$ rad/s^2, against -0.500 from the small-angle form.
- Small-angle period: $2\pi/\sqrt{5} = 2.81$ s.
- At $\theta = 0.5$ rad the two forms differ by 4%, so large swings run slow.

Worked example: a bead on a spinning hoop

- $L = \tfrac{1}{2} m R^2(\dot\theta^2 + \Omega^2 \sin^2\theta) + mgR\cos\theta$.
- Euler–Lagrange: $R\ddot\theta = \sin\theta\,(\Omega^2 R\cos\theta - g)$. Equilibria: $\theta = 0$, or $\cos\theta = g/(\Omega^2 R)$ when $\Omega^2 > g/R$.
- With $R = 0.20$ m, $\Omega = 10$ rad/s and $g = 10$ m/s^2: $\cos\theta = 10/20 = 0.5$, so the bead rides at $60°$.

Common errors

- Writing $L = T + V$.
- Treating r as constant when differentiating $mr^2\dot{\theta}$.
- Calling a coordinate cyclic when only its rate is missing from L (it must be the coordinate that is missing).

Common errors (2)

- Using the frequency formula at an unstable equilibrium.
- Adding constraint forces back into the Lagrangian.

Key terms

- **constraint**: A condition that limits how a system can move.
- **generalised coordinates**: Independent variables that fix a system's configuration.
- **Euler–Lagrange equation**: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0.$
- **cyclic coordinate**: A coordinate that does not appear in the Lagrangian; its momentum is conserved.

Check yourself

- For $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$, which momentum is conserved?
- A particle in a central force moves from r to $r/2$. What happens to $\dot{\theta}$?
- For the spinning hoop with $\Omega^2 < g/R$, where can the bead rest stably?