

Constraints, variational equations and cyclic coordinates

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A pendulum tension changes throughout a swing, but its angular equation can be derived without solving for tension. Derive Euler–Lagrange motion after imposing a holonomic constraint Use a cyclic coordinate to identify conserved canonical momentum Linearise a stable equilibrium and check the resulting frequency

Choose independent coordinates

Reasoning

A constraint removes an independent coordinate before you form the kinetic energy. For a fixed-length pendulum choose θ from the downward vertical: $x=l \sin\theta$ and $y=-l \cos\theta$. Differentiating both coordinates gives $v^2=l^2\dot{\theta}^2$, not $l^2 \sin^2\theta \dot{\theta}^2$. Thus $T=ml^2\dot{\theta}^2/2$ and $V=mgl(1-\cos\theta)$, with zero potential at the bottom. The fixed length is a holonomic constraint: it is a relation among coordinates and possibly time. A rolling velocity constraint needs its own analysis; do not automatically treat every constraint as a coordinate substitution.

Derive the equation

Reasoning

For ideal constraints and the usual conservative system, the stationary-action equation is $d/dt(\partial L/\partial \dot{q}) - \partial L/\partial q = 0$ with $L = T - V$. The variations vanish at the two time endpoints. A pendulum gives $\partial L/\partial \dot{\theta} = ml^2 \dot{\theta}$ and $\partial L/\partial \theta = -mgl \sin \theta$, so $ml^2 \ddot{\theta} + mgl \sin \theta = 0$. The momentum derivative acts on every time-dependent factor: for a varying radius, $d(mr^2 \dot{\theta})/dt$ contains $2mr \dot{r} \dot{\theta}$. The variational method does not mean mechanical energy is conserved in a time-dependent system.

Find cyclic momentum

Reasoning

For planar central motion, $L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$. The angle θ is cyclic because L has no explicit θ dependence, even though L depends on $\dot{\theta}$. Its canonical momentum $p_\theta = mr^2\dot{\theta}$ is constant. The radial equation is $m r\ddot{r} = mr\dot{\theta}^2 - dV/dr$. The first term is part of the coordinate acceleration, not a new outward real force in an inertial frame. Conservation of p_θ makes angular speed increase as r decreases; constant angular speed is a different, externally driven situation.

Linearise with conditions

Reasoning

Near a stable equilibrium q_0 , expand V to second order and use a constant local inertia M : $V \approx V(q_0) + V''(q_0)(q - q_0)^2/2$. Then the small-displacement frequency is $\sqrt{V''/M}$. For a pendulum $\sin\theta \approx \theta$ in radians, so $\omega = \sqrt{g/l}$; the approximation requires small amplitude. A negative V'' gives instability, not an oscillation with an imaginary measurable frequency. Check the full nonlinear equation first, then state the approximation and compare units: V''/M must have units of inverse time squared.

Worked reasoning 1

Reasoning

For a fixed pendulum length l , use θ from downward vertical: $x = l \sin \theta$, $y = -l \cos \theta$. Both Cartesian velocities give $T = m l^2 \dot{\theta}^2 / 2$ and $V = m g l (1 - \cos \theta)$. The Euler-Lagrange equation — is

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0.$$

Worked reasoning 3

Reasoning

Divide by $m\ell^2$ before linearising:

Worked relation 4

Reasoning

Apply this relation only under the conditions just stated.

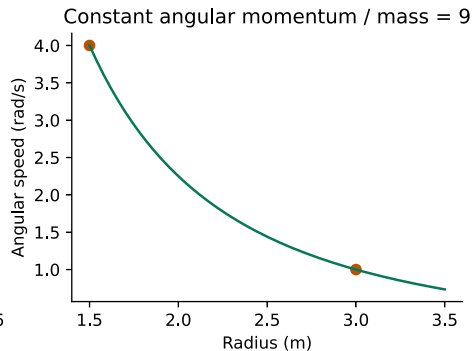
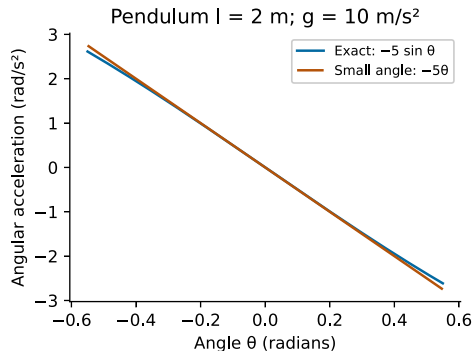
$$\ddot{\theta} + (g/l) \sin \theta = 0, \quad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

Worked reasoning 5

Reasoning

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.

Read the original diagram



- Original GRE teaching diagram: Constraints, variational equations and cyclic coordinates
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

A bead of mass m slides without friction on a horizontal radial slot rotating at prescribed constant angular speed Ω . Use radius r as coordinate to derive its equation and conserved energy-like quantity. Is its ordinary mechanical energy conserved? Explain. **[5]**

Independent transfer · reasoning

Reasoning

Prescribed rotation gives $L = m(\dot{r}^2 + \Omega^2 r^2)/2$. Euler-Lagrange gives $\ddot{r} = \Omega^2 r$. The time-independent reduced L has conserved $\dot{r} \partial L / \partial \dot{r} - L = m(\dot{r}^2 - \Omega^2 r^2)/2$. Mechanical energy is $m(\dot{r}^2 + \Omega^2 r^2)/2$ and is not generally conserved: the rotation drive supplies work through the constraint. Award L , equation, conserved quantity, mechanical-energy distinction and drive. The bead's angular momentum is not conserved.

Check assumptions and limits

Reasoning

A cyclic coordinate is absent from L itself; its velocity need not be absent. Do not discard the time derivative of a changing r^2 factor.

Keep the technical distinctions

Reasoning

holonomic constraint : A constraint expressible as a relation among coordinates and possibly time. cyclic coordinate : A coordinate absent explicitly from the Lagrangian, with conserved canonical momentum under the Euler–Lagrange equation.

Check yourself

- Define holonomic constraint and give its condition of use.
- Define cyclic coordinate and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.