Scattering, degeneracy and Pauli operators

GRE undergraduate preparation · local supported objectives

Scattering, degeneracy and Pauli operators

- Find reflection and **transmission 透射** at a potential step, using the **probability current 概率流**.
- Estimate **tunnelling 隧穿** through a barrier.
- Count the **degeneracy 简并度** of the 3D oscillator.
- Multiply **Pauli matrices** and find spin eigenvalues.

Steps and wave numbers

- A particle partly reflects even from a downward step, where nothing classical would turn it back.
- When $k_2 = k_1$ (no step), $r = 0$.

$$r = \frac{k_1 - k_2}{k_1 + k_2}, \qquad t = \frac{2k_1}{k_1 + k_2}$$

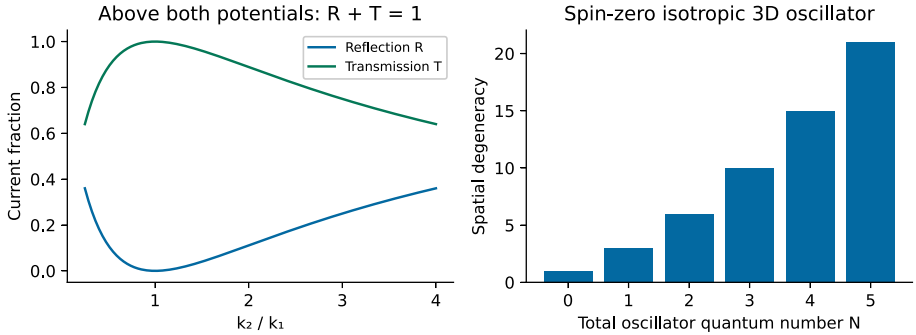
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Probability current

- Without the factor, $R + |t|^2$ does not add to 1.
- The transmission probability carries a factor k_2/k_1, because the **probability current** is proportional to k:

$$R = |r|^2, \qquad T = \frac{k_2}{k_1}|t|^2, \qquad R + T = 1$$

Figure: Probability current



- $R + T = 1$, with $R = 0$ when $k_2 = k_1$; level N has $(N + 1)(N + 2)/2$ states
- Read the axes and units first.

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Tunnelling through a barrier

- Doubling the width squares the (small) transmission.
- Alpha decay and the scanning **tunnelling** microscope both depend on this.

$$T \approx e^{-2\kappa a}$$

The 3D isotropic oscillator

- The number of ways to share N among three directions is $\frac{(N+1)(N+2)}{2}$: 1, 3, 6, 10, ... (spin ignored).
- $N = 2$: the arrangements of $(2,0,0)$ and $(1,1,0)$, six states.

Spin and the Pauli matrices

- Each squares to the identity, so its eigenvalues are ± 1. Spin components are $S_i = \frac{\hbar}{2}\sigma_i$, giving $\pm\hbar/2$: Stern and Gerlach's two spots.
- $\sigma_x\sigma_y = i\sigma_z$, cyclic in $x \to y \to z$; reversing the order flips the sign.
- The eigenvectors of σ_x are $(1, \pm 1)/\sqrt{2}$.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Worked example: a downward step

- $r = (1-3)/(1+3) = -\frac{1}{2}$ and $t = 2/(1+3) = \frac{1}{2}$.
- $R = \frac{1}{4}$ and $T = 3 \times \frac{1}{4} = \frac{3}{4}$, so $R + T = 1$.
- Using $|t|^2$ alone would give a total of $\frac{1}{2}$: probability would vanish.

Worked example: an electron tunnelling

- $\kappa = \sqrt{2(9.11 \times 10^{-31})(4.0 \times 1.6 \times 10^{-19})}/(1.05 \times 10^{-34}) = 1.0 \times 10^{10}$ m^{-1}.
- $2\kappa a = 2(1.0 \times 10^{10})(0.50 \times 10^{-9}) = 10$.
- $T \approx e^{-10} = 4.5 \times 10^{-5}$: about one electron in 22,000 gets through.

Common errors

- Forgetting the factor k_2/k_1 in the transmission.
- Saying a particle cannot reflect from a downward step.
- Counting the oscillator degeneracy as $N+1$ (that is the 2D answer).

Common errors (2)

- Writing $\sigma_y\sigma_x = i\sigma_z$; it is $-i\sigma_z$.
- Thinking a spin component can take any value between $-\hbar/2$ and $+\hbar/2$.

Key terms

- **spin**: The intrinsic angular momentum of a particle; for an electron, $\pm\hbar/2$ along any axis.
- **Pauli matrices**: The three 2 by 2 matrices that represent spin one-half.
- **transmission**: The fraction of particles that pass a step or barrier.
- **probability current**: The flow of probability per unit time, proportional to k for a plane wave.

Check yourself

- How many states does the $N = 3$ level of the 3D oscillator have (no spin)?
- What is $\sigma_y \sigma_x$?
- What is R when $k_2 = k_1$?