

Scattering, degeneracy and Pauli operators

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A quantum wave can reflect from a downward potential step even though a classical particle has no turning point there. Match travelling waves and fluxes across finite potential changes Count degenerate isotropic-oscillator states Evaluate Pauli products and distinguish anticommutation from equality

Choose the system and model

Reasoning

For constant potential V and energy $E > V$, the spatial solutions are travelling factors $\exp(\pm ikx)$, with $k = \sqrt{2m(E - V)}/\hbar$. At finite steps with the same particle mass, wavefunction and its first derivative are continuous. In a left-incident scattering problem with no incoming beam from the right, the far-right solution contains only the right-travelling factor Ae^{ikx} . If a finite well returns to the original external potential, the transmitted external wave number equals the incident one, even though the interior wave number differs. Exponentially decaying solutions describe $E < V$ regions, not every potential well.

Use the governing relation

Reasoning

For a step from V_1 to V_2 with both regions classically allowed, write incident-plus-reflected amplitude in region 1 and transmitted amplitude in region 2. Continuity gives $r=(k_1-k_2)/(k_1+k_2)$ and $t=2k_1/(k_1+k_2)$. Reflection probability is $R=|r|^2$; transmission is $T=(k_2/k_1)|t|^2$ because probability current depends on wave number. Thus $R+T=1$ for a lossless step. Do not add raw squared transmitted amplitude to R without its current factor. With $k_2=3k_1$, $R=1/4$ and $T=3/4$ despite a downward step.

Apply the conditions

Reasoning

A three-dimensional isotropic oscillator separates into x, y, z modes with nonnegative integers n_x, n_y, n_z . Its energy is $(N+3/2)\hbar\omega$, where $N=n_x+n_y+n_z$. For a spin-zero distinguishable single particle, the spatial degeneracy is the number of such triples: $(N+1)(N+2)/2$. This counts different assignments, not just different unordered partitions. For $N=2$, the six states are permutations of $(2,0,0)$ and $(1,1,0)$. For $N=3$, the degeneracy is 10. Additional spin or identical-particle constraints would change the counting problem.

Check the conclusion

Reasoning

Pauli matrices are $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Each squares to identity. Direct multiplication gives $\sigma_x \sigma_y = i \sigma_z$ and $\sigma_y \sigma_x = -i \sigma_z$, so distinct Pauli matrices anticommute. Cyclic products $x \rightarrow y \rightarrow z$ have positive i ; reversing order changes the sign. The general identity is $\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k$. Matrix order matters: $\sigma_x \sigma_z = -i \sigma_y$, not $i \sigma_y$ or simply σ_y . These dimensionless matrices become spin operators $S_i = \hbar \sigma_i / 2$, adding physical units and factors.

Worked reasoning 1

Reasoning

At a finite potential step, match wavefunction and derivative for the same particle mass.

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

$$r = \frac{k_1 - k_2}{k_1 + k_2}, \quad t = \frac{2k_1}{k_1 + k_2}.$$

Worked reasoning 3

Reasoning

Transmission probability
 $r = -1/2$, $t = 1/2$.

uses current, not amplitude alone. For $k_2 = 3k_1$,

Worked relation 4

Reasoning

Apply this relation only under the conditions just stated.

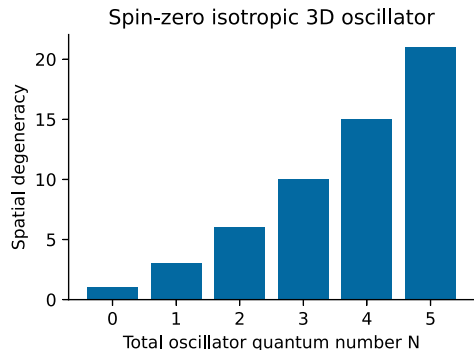
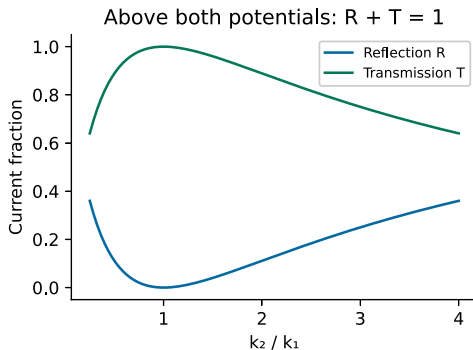
$$R = |r|^2 = 1/4, \quad T = (k_2/k_1)|t|^2 = 3/4, \quad R + T = 1.$$

Worked reasoning 5

Reasoning

Reflection occurs even for a downward step. Its phase sign disappears when taking probability.

Read the original diagram



- Original GRE teaching diagram: Scattering, degeneracy and Pauli operators
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

A spin-half particle has state $(\sqrt{3}, i)^T/2$ in the z-spin basis. Check normalisation and find probabilities for S_z and S_x outcomes. **[5]**

Independent transfer · reasoning

Reasoning

Norm is $(3 + 1)/4 = 1$. S_z gives $+\hbar/2$ with probability $3/4$ and $-\hbar/2$ with probability $1/4$. The x eigenstates are $(1, \pm 1)^T / \sqrt{2}$. Their amplitudes are $(\sqrt{3} \pm i)/(2\sqrt{2})$, each with squared modulus $1/2$. Award norm, z results/probabilities, x eigenbasis, projections and x probabilities. Complex conjugation is required in the inner product; the imaginary relative phase matters.

Check assumptions and limits

Reasoning

Transmission probability needs the wave-number current ratio. Oscillator degeneracy counts ordered mode triples. Do not reverse Pauli matrix order without changing its sign.

Keep the technical distinctions

Reasoning

reflection coefficient : Reflected probability-current fraction relative to incident current. degeneracy : Number of independent states sharing the specified energy.

Check yourself

- Define reflection coefficient and give its condition of use.
- Define degeneracy and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.