Counting statistics, uncertainty and dimensional models

GRE undergraduate preparation · local supported objectives

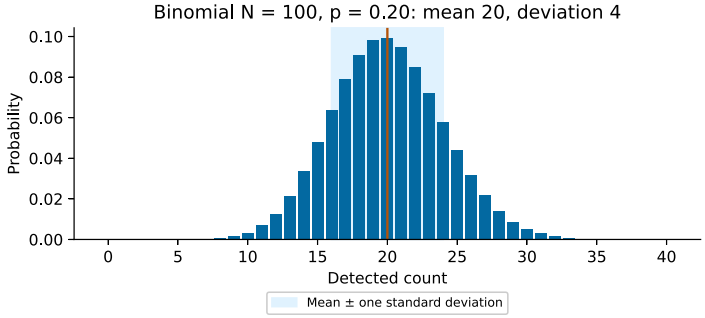
Counting statistics, uncertainty and dimensional models

- Use the binomial and Poisson distributions and their **standard deviation** 标准差.
- Propagate uncertainties through formulas.
- Separate random error from bias, and use the **standard error** 标准误差 of a mean.
- Use **dimensional analysis** 量纲分析 to find how a quantity depends on others.

The binomial distribution

- Mean Np, **variance 方差** $Np(1-p)$, **standard deviation** $\sqrt{Np(1-p)}$.
- A detector with 20% efficiency that receives 100 photons records 20 ± 4, not exactly 20.

Figure: The binomial distribution



- Identify the axes, units, directions and the stated model.

The Poisson distribution

- Counts of radioactive decays, photons or cosmic rays are Poisson: N counts have an uncertainty of $\sqrt{N}$.
- $N = 200$, $p = 0.05$: exact $\sigma = \sqrt{9.5} = 3.08$; Poisson $\sqrt{10} = 3.16$, close because p is small.

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad \sigma = \sqrt{\lambda}$$

Propagating uncertainty

- For $y = x^a$, the relative uncertainty of y is $|a|$ times that of x.
- For products and quotients, add relative uncertainties in quadrature. Adding them in a straight line gives only a worst case.

$$\sigma_y^2 \approx \sum_i \left(\frac{\partial y}{\partial x_i} \right)^2 \sigma_i^2$$

Precision, accuracy and averaging

- A bias shared by every reading (a badly calibrated meter) is not reduced by averaging.
- To halve the **standard error**, you need four times as many readings.

Dimensional analysis

- $[G] = \mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$, $[\hbar] = \mathrm{ML}^2\mathrm{T}^{-1}$, $[c] = \mathrm{LT}^{-1}$.
- A pendulum's period can depend only on L and g: $T \propto \sqrt{L/g}$, and mass cannot enter.
- Dimensions fix the powers but not pure numbers such as 2π.

Worked example: counting uncertainties

- Counts: $400 \pm \sqrt{400} = 400 \pm 20$, a 5% uncertainty. For 1%, you need $1/\sqrt{N} = 0.01$: 10,000 counts.
- Kinetic energy with exact mass: $2 \times 4\% = 8\%$.
- With 3% in m and 2% in v: $\sqrt{3^2 + (2 \times 2)^2} = 5\%$.

Worked example: the Planck length

- M: $-a + b = 0$. L: $3a + 2b + d = 1$. T: $-2a - b - d = 0$.
- So $a = b = \frac{1}{2}$ and $d = -\frac{3}{2}$: $\ell = \sqrt{G\hbar/c^3}$.
- $\ell = \sqrt{(6.67 \times 10^{-11})(1.05 \times 10^{-34})/(3.0 \times 10^{8})^3} = 1.6 \times 10^{-35}$ m.

Common errors

- Expecting a detector of 20% efficiency to record exactly 20% of events.
- Using the Poisson $\sqrt{\lambda}$ when p is not small.
- Adding independent uncertainties linearly.

Common errors (2)

- Forgetting to multiply the relative uncertainty by the power.
- Thinking more readings remove a calibration bias.

Key terms

- **Poisson distribution**: The distribution of counts of independent random events, with **variance** equal to its mean.
- **standard deviation**: The typical spread of values about the mean; the square root of the variance.
- **standard error**: The uncertainty of a mean, $\sigma/\sqrt{n}$ for n independent readings.
- **dimensional analysis**: Matching the powers of mass, length and time on both sides of an equation.

Check yourself

- A counter records 900 counts. What is the uncertainty?
- A quantity depends on x^3, and x has a 2% uncertainty. What is the uncertainty of the quantity?
- How many readings are needed to make the **standard error** a third of that from one reading?