

Speed distributions and statistical ensembles

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A gas can have zero mean velocity while its particles have substantial average speed. Opposite directions cancel in a vector mean; speed has no sign. Normalise continuous speed densities and interpret their units Distinguish vector means, speed means and most probable speeds Use Boltzmann weights with degeneracy and a partition function

Choose the system and model

Reasoning

For a continuous speed probability density $f(v)$, $f(v)dv$ approximates probability in a narrow interval and $\int_{0\text{ to } \infty} f(v)dv=1$. The density has inverse-speed units and is not itself a probability. If $P(v)dv$ counts particles in that interval, its area is the particle count N and $f=P/N$. A trapezoid rising over width v_0 , remaining flat over width $2v_0$, then falling over width v_0 has area $3av_0$ for height a . Normalise by area before computing a mean or interpreting a plotted height.

Use the governing relation

Reasoning

In an isotropic equilibrium gas without bulk flow, each velocity-component distribution is symmetric, so $\langle \mathbf{v}_{\text{vector}} \rangle = 0$. Speed $v = |\mathbf{v}_{\text{vector}}|$ is nonnegative and has positive mean. For a classical ideal gas, the Maxwell speed density is proportional to $v^2 \exp[-mv^2/(2kBT)]$. Its mode is $\sqrt{2kBT/m}$, its mean is $\sqrt{8kBT/(\pi m)}$ and its rms speed is $\sqrt{3kBT/m}$. These are three different quantities. The v^2 factor makes speed density zero at $v=0$, even though the velocity-vector density is largest at the zero vector.

Apply the conditions

Reasoning

For a continuous distribution, the probability of exactly one specified speed is zero; nonzero probabilities refer to intervals. This statement does not imply there are no particles with arbitrarily small speeds, or that a finite-resolution detector cannot record a zero bin. Temperature changes the scale of the Maxwell distribution: characteristic speeds are proportional to $\sqrt{T/m}$. Doubling temperature does not double the speed, and heavier particles are slower on average at the same temperature.

Check the conclusion

Reasoning

In a canonical ensemble at temperature T , a state of energy E_i has weight $\exp(-E_i/(kBT))$. Divide by partition function $Z = \sum \exp(-E_i/(kBT))$ to obtain probabilities. If an energy level has degeneracy g_i , its level probability is $g_i \exp(-E_i/(kBT))/Z$. Equal energy per state does not imply equal probability per level when degeneracies differ. For a ground level of degeneracy 1 and an excited level of degeneracy 2 at $\Delta = kBT \ln 2$, the excited total weight is $2e^{(-\ln 2)} = 1$: the two levels each have probability $1/2$. Classical Maxwell-Boltzmann assumptions differ from Bose-Einstein and Fermi-Dirac quantum occupations.

Worked reasoning 1

Reasoning

A canonical ensemble
gap Δ .

has ground degeneracy 1 and excited degeneracy g at

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

$$Z = 1 + ge^{-\Delta/(k_B T)}, \quad P_e = \frac{ge^{-\Delta/(k_B T)}}{Z}.$$

Worked reasoning 3

Reasoning

With $g = 4$ and $\Delta = k_B T \ln 2$,

Worked relation 4

Reasoning

Apply this relation only under the conditions just stated.

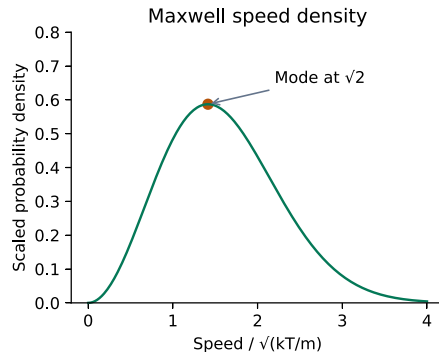
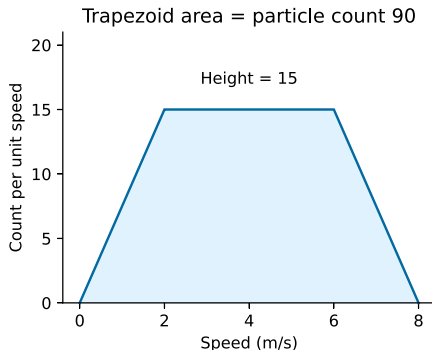
$$P_e = \frac{4e^{-\ln 2}}{1 + 4e^{-\ln 2}} = \frac{2}{3}.$$

Worked reasoning 5

Reasoning

This is probability for the entire excited level. Each of its four states has probability $1/6$.

Read the original diagram



- Original GRE teaching diagram: Speed distributions and statistical ensembles
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

Two ideal gases at the same temperature have particle masses m and $4m$. Compare their most probable, mean and rms speeds. Compare their mean translational kinetic energies and explain why zero mean velocity does not mean zero energy. **[4]**

Independent transfer · reasoning

Reasoning

All three characteristic speeds scale as $\sqrt{T/m}$, so heavier/light speed ratio is 1/2. Mean translational energy is $3k_B T/2$ for either gas in the classical regime. Random velocity directions cancel in $\langle \mathbf{v} \rangle$, but $\langle v^2 \rangle$ is positive. Award speed ratios, regime, energy equality and vector distinction.

Check assumptions and limits

Reasoning

A density height is not a probability. Mean velocity, mean speed and mode differ; include degeneracy before normalising level probabilities.

Keep the technical distinctions

Reasoning

probability density : Probability per unit of a continuous variable, whose integral gives interval probability. partition function : Sum of statistical weights used to normalise equilibrium state probabilities.

Check yourself

- Define probability density and give its condition of use.
- Define partition function and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.