Fourier symmetry and wave superposition

GRE undergraduate preparation · local supported objectives

Fourier symmetry and wave superposition

- Find Fourier **coefficients** 系数 and the mean value.
- Use symmetry (even and odd functions) to kill terms before integrating.
- Connect smoothness with how fast the coefficients fall.
- Add coherent waves by amplitude.

Fourier coefficients

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.
- The mean value of f is $a_0/2$, not a_0.
- The terms with $n > 1$ are the **harmonics** 谐波 of the fundamental frequency.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Symmetry kills terms

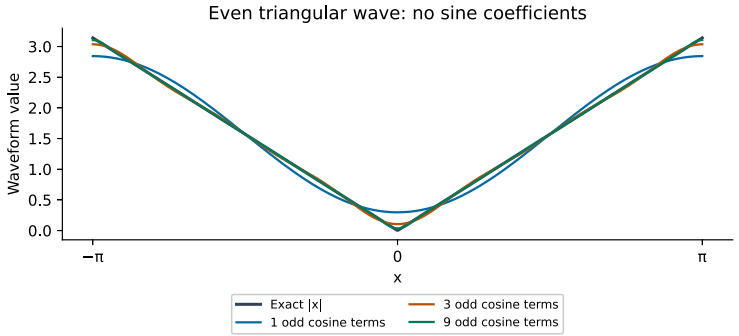
- An **even function 偶函数**, $f(-x) = f(x)$, has only cosines: every $b_n = 0$.
- An **odd function 奇函数**, $f(-x) = -f(x)$, has only sines: the mean and every a_n are zero.
- Symmetry depends on where you put the origin. Shifting a signal can mix sines and cosines without changing which frequencies it contains.

The triangle wave

- The coefficients fall as $1/n^2$, because the function is continuous but has corners.
- $f(x) = |x|$ on $[-\pi, \pi]$, repeated, is even, so it has only cosines.

$$a_n = \frac{2\left[(-1)^n - 1\right]}{\pi n^2} = -\frac{4}{\pi n^2} \text{ for odd } n, \quad 0 \text{ for even } n$$

Figure: The triangle wave



- An **even function** needs only cosine terms; more terms sharpen the corner
- Read the axes and units first.

The square wave and the Gibbs overshoot

- A jump makes the coefficients fall only as $1/n$, more slowly than for the triangle wave.
- Near each jump, the partial sums overshoot by about 9% of the jump, however many terms you add: the **Gibbs phenomenon 吉布斯现象**.
- Smoother functions need fewer terms: a corner gives $1/n^2$, a jump gives $1/n$.

Coherent superposition

- At $\varphi = \pi/2$, $I = 2I_0$; at $\varphi = \pi$, zero.
- For incoherent light the cross term averages to zero, so the intensity is always $2I_0$.

$$I = 2I_0(1 + \cos\varphi)$$

Worked example: partial sums of the triangle wave

- One term: $\dfrac{\pi}{2} - \dfrac{4}{\pi}\cos 0 = 1.571 - 1.273 = 0.298.$
- Two odd terms: $1.571 - 1.273\left(1 + \tfrac{1}{9}\right) = 0.156.$
- The error falls slowly towards zero as more odd terms are added, as the $1/n^2$ coefficients predict.

Worked example: the square wave at its peak

- $\frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2}\right) = 1.273\left(1 - \frac{1}{3} + \frac{1}{5}\right)$.
- $= 1.273 \times 0.867 = 1.10$, already close to 1.
- The first term alone has amplitude $4/\pi = 1.27$, larger than the square wave itself.

Common errors

- Taking the mean as a_0 instead of $a_0/2$.
- Expecting sine terms in an **even function**.
- Forgetting that symmetry depends on the chosen origin.

Common errors (2)

- Expecting the Gibbs overshoot to vanish with more terms.
- Adding intensities for coherent waves.

Key terms

- **Fourier series**: A sum of sines and cosines that represents a periodic function.
- **coefficient**: The number that multiplies one term of a series.
- **harmonics**: Frequencies that are whole-number multiples of the fundamental.
- **even function**: A function with $f(-x) = f(x)$.

Check yourself

- Which terms appear in the **Fourier series** of x^3 on $[-\pi, \pi]$?
- A Fourier series has $a_0 = 4$. What is the mean of the function?
- Whose coefficients fall faster, the triangle wave's or the square wave's?