

Fourier symmetry and wave superposition

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

A waveform can contain many harmonics while having no sine terms at all. Reflection symmetry tells you which coefficients vanish before integration. Use orthogonality to identify Fourier coefficients. Exploit even and odd parity without discarding the constant term. Relate component amplitudes to interference and physical waveforms.

Choose the system and model

Reasoning

For a sufficiently regular real 2π -periodic function, write $f(x) = a_0/2 + \sum [a_n \cos(nx) + b_n \sin(nx)]$. The coefficients are $a_n = (1/\pi) \int_{-\pi}^{\pi} f(x) \cos(nx) dx$ and $b_n = (1/\pi) \int_{-\pi}^{\pi} f(x) \sin(nx) dx$, with a_0 using $n=0$. Cosines and sines are orthogonal on a full period. The mean is $a_0/2$, not a_0 ; retain the constant term even when all nonzero-frequency coefficients of one family vanish.

Use the governing relation

Reasoning

If f is even about the chosen origin, $f(x)\sin(nx)$ is odd and its symmetric integral is zero, so all b_n vanish. If f is odd, its mean and all a_n vanish. Symmetry depends on the origin: shifting the same physical signal can mix sine and cosine coefficients while leaving its harmonic frequencies unchanged. A nonnegative triangular waveform symmetric about $x=0$ can therefore have cosine harmonics and a nonzero mean without any sine harmonics.

Apply the conditions

Reasoning

For the 2π -periodic extension of $f(x)=|x|$ on $[-\pi,\pi]$, the mean is $\pi/2$. Integrating $x \cos(nx)$ by parts on $[0,\pi]$ gives $a_n=2[(-1)^{n-1}]/(\pi n^2)$, so even- n cosine coefficients are zero and odd- n coefficients are $-4/(\pi n^2)$. All sine coefficients are zero. The $1/n^2$ decay reflects a continuous function with a slope discontinuity. A jump discontinuity often gives slower coefficient decay and partial-sum overshoot; do not infer the same convergence behaviour for every waveform.

Check the conclusion

Reasoning

A Fourier sum adds amplitudes with their phases. For equal coherent monochromatic amplitudes A meeting with phase difference ϕ , resultant squared amplitude is $2A^2(1+\cos\phi)$, so intensity is $2I_0(1+\cos\phi)$. It is $4I_0$ in phase and zero at $\phi=\pi$. Incoherent averaging removes the cross term, yielding $2I_0$. Harmonics at different frequencies can construct a shape over time; their instantaneous sum is not the sum of their individual intensities. Identify whether the question concerns a waveform, time-average power or coherent interference.

Worked reasoning 1

Reasoning

For a real periodic function, use $f = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$. Even symmetry makes all sine coefficients zero because the integrand is odd. For $f(x) = |x|$ on $[-\pi, \pi]$,

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

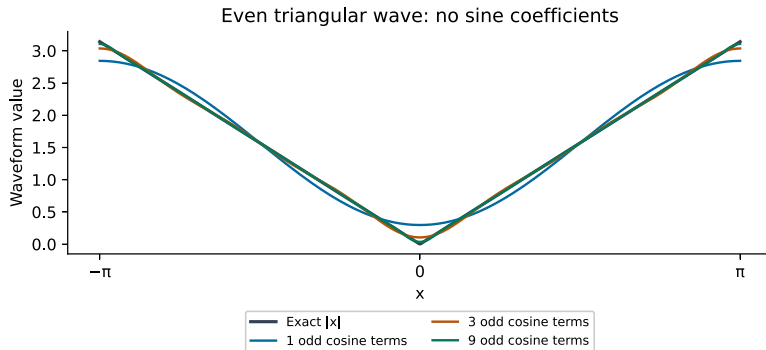
$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) dx = \frac{2[(-1)^n - 1]}{\pi n^2}.$$

Worked reasoning 3

Reasoning

Integration by parts supplies the second expression. The mean is $a_0/2 = \pi/2$; only odd cosine harmonics survive. A cusp does not justify dropping the constant term.

Read the original diagram



- Original GRE teaching diagram: Fourier symmetry and wave superposition
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

Shift this waveform by $\pi/2$: $g(x) = f(x - \pi/2)$. Write its first nonconstant harmonic and its mean. Explain why its sine coefficient need not vanish, although the unshifted waveform is even. **[4]**

Independent transfer · reasoning

Reasoning

Substitute the shift in $f \approx \pi/2 - (4/\pi) \cos x$: $g \approx \pi/2 - (4/\pi) \sin x$. Its mean remains $\pi/2$. Evenness depends on the chosen origin; g is not even about zero. Frequencies stay unchanged. Award substitution, harmonic/sign, mean and origin/frequency interpretation.

Check assumptions and limits

Reasoning

Even symmetry removes sine coefficients, not the constant term or every cosine harmonic. Check the symmetry origin and whether fields are coherent before adding intensities.

Keep the technical distinctions

Reasoning

orthogonality : A zero integral of the product of distinct basis functions over the specified interval. Fourier coefficient : Weight of a sine, cosine or constant basis component in a periodic expansion.

Check yourself

- Define orthogonality and give its condition of use.
- Define Fourier coefficient and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.