Maxwell waves, energy flow and boundary conditions

GRE undergraduate preparation · local supported objectives

Maxwell waves, energy flow and boundary conditions

- Use the displacement current in Ampère's law.
- Find the speed, impedance and field ratio of waves in a medium.
- Get directions from $\mathbf{E} \times \mathbf{B}$, the **Poynting vector 坡印廷矢量**.
- Apply **boundary conditions 边界条件** at surfaces.

Displacement current

- Between the plates, $I_d = \varepsilon_0 A\, dE/dt$ equals the current in the wires.
- It is a rate of change of electric flux, not a flow of charge.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(I + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

Waves in a medium

- For a non-magnetic medium, $v = c/\sqrt{\varepsilon_r}$: $\varepsilon_r = 9$ gives $v = 1.0 \times 10^8$ m/s and $n = 3$.
- Use the **permittivity** at the wave's frequency. Water has $\varepsilon_r \approx 80$ for static fields, but $n = 1.33$ for light.

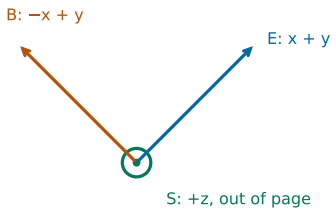
$$v = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad n = \frac{c}{v} = \sqrt{\mu_r \varepsilon_r}, \qquad \frac{E}{B} = v$$

Directions of E, B and S

- The **Poynting vector** $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ points along the energy flow, which is the direction of travel.
- A wave in $kz - \omega t$ travels towards $+z$; one in $kz + \omega t$ travels towards $-z$.
- Travelling along $+z$ with $\mathbf{E} \parallel \hat{x} + \hat{y}$: $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{z} \times \hat{y} = -\hat{x}$, so $\mathbf{B} \parallel -\hat{x} + \hat{y}$.

Figure: Directions of E, B and S

E × B points in the propagation direction



- Identify the axes, units, directions and the stated model.

Energy and intensity

- Light carries momentum too, so it exerts **radiation pressure** 辐射压: I/c on an absorbing surface, $2I/c$ on a perfect mirror.
- **Intensity** from a point source falls as $1/r^2$, so E_0 falls as $1/r$.

$$I = \langle S \rangle = \tfrac{1}{2} c \varepsilon_0 E_0^2$$

Boundary conditions

- Normal $\mathbf{B}$ is continuous ($\nabla \cdot \mathbf{B} = 0$); tangential $\mathbf{E}$ is continuous.
- Normal $\mathbf{D}$ jumps by the free surface charge; tangential $\mathbf{H}$ jumps by the free surface current.
- At an ideal superconductor ($B = 0$ inside), the field just outside has no normal component: it runs along the surface. At a perfect conductor, tangential E is zero.

Worked example: a charging capacitor

- $dE/dt = \dfrac{I}{\varepsilon_0 A} = \dfrac{2.0}{(8.85 \times 10^{-12})(0.010)} = 2.3 \times 10^{13}$ V/(m s).
- Displacement current: $\varepsilon_0 A\, dE/dt = 2.0$ A, equal to the wire current.
- So a loop round the gap, 10 cm from the axis (outside circular plates of radius 5.6 cm), has $B = \mu_0 I/(2\pi r) = 4.0\ \mu$T, the same as round the wire.

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Worked example: a laser beam

- **Intensity**: $I = 10^{-3}/10^{-6} = 1.0 \times 10^{3}$ W/m^2.
- $E_0 = \sqrt{2I/(c\varepsilon_0)} = \sqrt{2000/(2.66 \times 10^{-3})} = 870$ V/m and $B_0 = E_0/c = 2.9\ \mu$T.
- Pressure on an absorber: $I/c = 3.3 \times 10^{-6}$ Pa.

Common errors

- Treating the displacement current as moving charge.
- Using the static **permittivity** for light.
- Getting $\mathbf{B}$ the wrong way round; check that $\mathbf{E} \times \mathbf{B}$ points along the travel.

Common errors (2)

- Forgetting the factor $\frac{1}{2}$ in the average **intensity**.
- Saying the field at a superconductor's surface must be zero; only its normal part is.

Key terms

- **displacement current**: $\varepsilon_0 \, d\Phi_E/dt$, the term that completes Ampère's law.
- **Poynting vector**: $\mathbf{E} \times \mathbf{B}/\mu_0$, the energy flow per unit area.
- **boundary conditions**: Rules for how fields change across a surface between two media.
- **permittivity**: How strongly a medium reduces an electric field; $\varepsilon = \varepsilon_r \varepsilon_0$.

Check yourself

- A non-magnetic medium has $\varepsilon_r = 4$. What are v and n?
- A wave has $E \propto \cos(kz + \omega t)$. Which way does it travel?
- Is the normal component of $\mathbf{B}$ continuous at a boundary?