Hamiltonian mechanics and astrophysical scaling

GRE undergraduate preparation · local supported objectives

Hamiltonian mechanics and astrophysical scaling

- Build the **Hamiltonian 哈密顿量** from a Lagrangian and use Hamilton's equations.
- Know when the Hamiltonian is conserved and when it equals the energy.
- Scale orbital speed, period and angular momentum with radius.
- Use $L = 4\pi R^2 \sigma T^4$, the inverse-square law and Wien's law.

From Lagrangian to Hamiltonian

- Oscillator: $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$ gives $p = m\dot{q}$ and $H = \dfrac{p^2}{2m} + \frac{1}{2}kq^2$.
- In a magnetic field, the **canonical momentum** $m\mathbf{v} + q\mathbf{A}$ is not the same as $m\mathbf{v}$.

$$H(q, p) = p\dot{q} - L$$

Hamilton's equations

- Oscillator: $\dot{q} = p/m$ and $\dot{p} = -kq$, so $\ddot{q} = -(k/m)q$, as Newton gives.
- If H does not contain q, then p is conserved.
- If H has no explicit time dependence, H is conserved. It equals $T + V$ when the constraints do not depend on time and V does not depend on velocity; otherwise it may not.

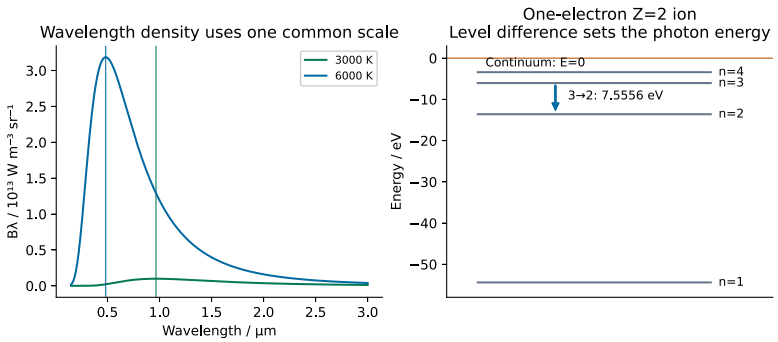
$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

Circular orbits

- $T^2 \propto r^3$ is **Kepler's third law** 开普勒第三定律.
- Total energy is $-GMm/2r$: a lower orbit is faster but has less energy.

$$v = \sqrt{\frac{GM}{r}}, \qquad T^2 = \frac{4\pi^2 r^3}{GM}, \qquad L = m\sqrt{GMr}$$

Figure: Circular orbits



- Doubling the temperature halves the peak wavelength; for a $Z = 2$ ion every level is $Z^2 = 4$ times the hydrogen value
- Read the axes and units first.

Stars: luminosity and flux

- Twice the radius at the same temperature: four times the luminosity. Twice the temperature: sixteen times.
- Twice as far away: a quarter of the **flux**. That is why **parallax** 视差 distances matter.

$$L = 4\pi R^2 \sigma T^4, \qquad F = \frac{L}{4\pi d^2}$$

Wien's law and spectra

- The Sun (5800 K) peaks near 500 nm; a 3000 K red dwarf peaks near 970 nm, in the infrared.
- Spectral lines add composition: a one-electron ion with $Z = 2$ has every level $Z^2 = 4$ times deeper than hydrogen, so its $3 \to 2$ photon is $4 \times 1.89 = 7.56$ eV.

Worked example: a higher satellite

- Speed: $v \propto r^{-1/2}$, so it moves at $1/3$ of the speed.
- Angular momentum: $L \propto r^{1/2}$, so it has 3 times as much.
- Period: $T \propto r^{3/2}$, so its year is $9^{3/2} = 27$ times longer.

Worked example: sunlight at the Earth

- **Flux**: $F = \dfrac{3.8 \times 10^{26}}{4\pi (1.5 \times 10^{11})^2} = 1.3 \times 10^{3}$ W/m^2, close to the measured 1361 W/m^2.
- Peak wavelength: $\lambda_{\max} = 2.9 \times 10^{-3} / 5800 = 5.0 \times 10^{-7}$ m, green light.
- A star of the same luminosity but 3 times farther away would give $1/9$ of this flux.

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Common errors

- Writing $H = T + V$ for every system, including time-dependent constraints.
- Using the same power of r for speed, period and angular momentum.
- Confusing luminosity (power given out) with **flux** (power received per square metre).

Common errors (2)

- Forgetting the fourth power of temperature in L.
- Using degrees Celsius in Wien's law.

Key terms

- **luminosity**: The total power a star gives out.
- **Hamiltonian**: The function of coordinates and momenta that generates the equations of motion.
- **canonical momentum**: $p = \partial L / \partial \dot{q}$, the momentum paired with a coordinate.
- **Kepler's third law**: For orbits round one body, T^2 is proportional to r^3.

Check yourself

- A star has twice the Sun's radius and the same surface temperature. How does its luminosity compare?
- Two stars have the same luminosity; one is 3 times farther away. How do their fluxes compare?
- For a free particle, $H = p^2/2m$. Which quantity is conserved, and why?