

Hamiltonian mechanics and astrophysical scaling

GRE undergraduate preparation · local supported objectives

Choose the physical question

Reasoning

Energy methods connect a spring oscillator and an orbit, but their coordinates and physical assumptions differ. Construct a simple Hamiltonian Use canonical equations Apply gravitational and radiative scaling

Choose the system and model

Reasoning

The canonical momentum is $p = \partial L / \partial \dot{q}$. The Legendre transform $H = p \dot{q} - L$ gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

Use the governing relation

Reasoning

Hamilton's equations are $\dot{q} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial q$. For an ordinary oscillator $H = p^2 / (2m) + kq^2 / 2$, these reproduce Newton's equation.

Apply the conditions

Reasoning

For a circular gravitational orbit $v^2 = GM/r$ and $\text{period}^2 = 4\pi^2 r^3 / (GM)$. The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass m , angular momentum magnitude is $mr v = m \sqrt{GM r}$, so identical satellites have L proportional to $\sqrt{r}$. A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

Check the conclusion

Reasoning

Luminosity and received flux obey $F=L/(4\pi d^2)$ for isotropic radiation. A blackbody has $L=4\pi R^2\sigma T^4$; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

Worked reasoning 1

Reasoning

For a regular Lagrangian $L = m\dot{q}^2/2 - kq^2/2$, canonical momentum is

Worked relation 2

Reasoning

Apply this relation only under the conditions just stated.

$$p = \partial L / \partial \dot{q} = m\dot{q}, \quad \dot{q} = p/m.$$

Worked reasoning 3

Reasoning

Substitute the velocity into the Legendre transform:

Worked relation 4

Reasoning

Apply this relation only under the conditions just stated.

$$H = p\dot{q} - L = p^2/(2m) + kq^2/2.$$

Worked relation 6

Reasoning

Apply this relation only under the conditions just stated.

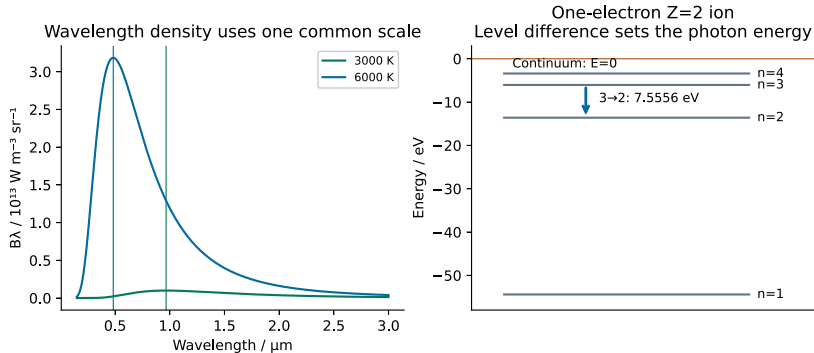
$$\dot{q} = \partial H / \partial p = p/m, \quad \dot{p} = -\partial H / \partial q = -kq.$$

Worked reasoning 7

Reasoning

Combining these gives $\ddot{q} = -(k/m)q$. The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.

Read the original diagram



- Original GRE teaching diagram: Hamiltonian mechanics and astrophysical scaling
- Identify the axes, units, directions and the stated model.

Independent transfer · attempt

Reasoning

An isotropically radiating blackbody star has twice the radius and half the temperature of another. It is three times farther from the observer. Find the luminosity ratio and received-flux ratio, ignoring absorption. Separately, derive Hamilton's equations for $H = p^2/(2m) + aq^4$ with a positive. **[5]**

Independent transfer · reasoning

Reasoning

$L \propto R^2 T^4$ gives $L_2/L_1 = 2^2(1/2)^4 = 1/4$. $F \propto L/d^2$ gives $F_2/F_1 = (1/4)/3^2 = 1/36$. Hamilton's equations give $\dot{q} = p/m$ and $\dot{p} = -4aq^3$. These describe an amplitude-dependent nonlinear oscillator; its restoring force is cubic, not a Hooke-law force with constant nonzero small-oscillation stiffness at zero. Award each ratio, each equation and the nonlinear interpretation.

Check assumptions and limits

Reasoning

A Hamiltonian equals total mechanical energy only under the relevant system assumptions; do not infer this universally from its name.

Keep the technical distinctions

Reasoning

canonical momentum : The derivative of the Lagrangian with respect to a generalised velocity. luminosity : Total emitted power, distinct from flux at an observer.

Check yourself

- Define canonical momentum and give its condition of use.
- Define luminosity and give its condition of use.
- Redo the original transfer with a changed parameter; check a limiting case.