Classical mechanics

GRE undergraduate preparation · local supported objectives

Classical mechanics

- Treat Newton's second law as a vector law, with a diagram and sensible axes.
- Pick the conservation law whose condition actually holds.
- Find small-oscillation frequencies from the shape of the potential.
- Use the **Lagrangian** or **Hamiltonian** when constraints make forces awkward.

Newton's second law in components

- Block on a frictionless incline of angle θ: along the slope $a = g\sin\theta$; across it, $N = mg\cos\theta$.
- Uniform circular motion: the net force points to the centre and has size mv^2/r. It is not an extra force; it is the sum of the real ones.
- Constant speed does not mean zero acceleration. A turning velocity needs a net force.

Choosing a conservation law

- **Momentum** is conserved when the net external impulse is zero, for example during a short collision.
- **Mechanical energy** is conserved when no non-conservative force (friction, a sticking collision) does work.
- **Angular momentum** 角动量 about a point is conserved when the net external torque about that point is zero.

Central forces and orbits

- With $L/m = r^2\omega = 9\ \mathrm{m}^2/\mathrm{s}$, the angular speed is 4 rad/s at 1.5 m and 1 rad/s at 3 m.
- This is Kepler's second law: equal areas in equal times.
- Radial motion can be studied with an effective potential, which adds a centrifugal term to $U(r)$:

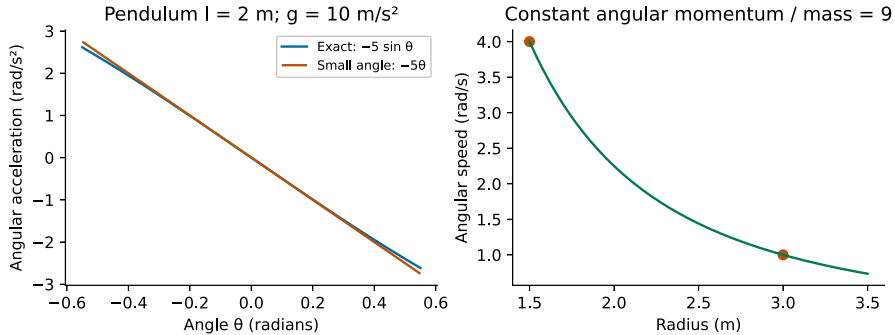
$$U_{\mathrm{eff}}(r) = U(r) + \frac{L^2}{2mr^2}$$

Small oscillations

- Spring: $\omega = \sqrt{k/m}$. Doubling the mass divides ω by $\sqrt{2}$, not by 2.
- Simple pendulum: for small angles $\sin\theta \approx \theta$, so $\ddot{\theta} = -(g/\ell)\theta$ and $\omega = \sqrt{g/\ell}$. A 2 m pendulum has $\ddot{\theta} \approx -4.9\,\theta$.
- A minimum is stable only if $U''(x_0) > 0$. At a maximum, a small push grows instead of oscillating.

$$U(x) \approx U(x_0) + \tfrac{1}{2}U''(x_0)(x - x_0)^2, \qquad \omega = \sqrt{\frac{U''(x_0)}{m}}$$

Figure: Small oscillations



- Small angles make the pendulum linear ($\ddot{\theta} \approx -5\theta$); constant angular momentum makes ω fall as $1/r^2$
- Read the axes and units first.

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Lagrangian and Hamiltonian mechanics

- If L does not contain a coordinate q (a cyclic coordinate), its momentum $p = \partial L/\partial \dot{q}$ is conserved.
- The **Hamiltonian 哈密顿量** $H = \sum p\dot{q} - L$ equals $T + V$ when the constraints do not depend on time. Hamilton's equations are $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$.
- Constraint forces (a normal force, a string tension) never appear, which is why this method suits beads, rails and pendulums.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Worked example: a spring by Lagrange

- Angular frequency: $\omega = \sqrt{k/m} = \sqrt{18/2.0} = 3.0$ rad/s.
- Period: $T = 2\pi/\omega = 2\pi/3.0 = 2.09$ s.
- Check: k/m has units of s^{-2}, and positive stiffness gives oscillation, not runaway motion.

$$L = \tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} k x^2 \quad \Rightarrow \quad m\ddot{x} + kx = 0$$

Worked example: rolling down a slope

- Energy: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ with $\omega = v/R$, so $mgh = \frac{3}{4}mv^2$ and $v = \sqrt{4gh/3} = \sqrt{4 \times 9.8 \times 1.2/3} = 3.96$ m/s.
- Acceleration: $a = \dfrac{g\sin\theta}{1 + I/(mR^2)} = \frac{2}{3}g\sin 30^\circ = 3.27$ m/s^2.
- Static friction makes it roll but does no work, because the contact point does not slide. So energy conservation still holds.

Common errors

- Adding "centripetal force" as an extra force on the diagram. It is the net inward force.
- Using kinetic-energy conservation in a collision just because momentum is conserved.
- Forgetting that angular momentum depends on the point chosen; choose the point where the torque is zero.

Common errors (2)

- Using $\omega = \sqrt{g/\ell}$ for large swings. It holds only while $\sin\theta \approx \theta$.
- Writing $L = T + V$. The **Lagrangian** is $T - V$; the **Hamiltonian** is usually $T + V$.

Key terms

- **centripetal acceleration**: The acceleration toward the centre of a circular path, v^2/r.
- **free-body diagram**: A sketch of one object showing every force acting on it.
- **angular momentum**: The moment of linear momentum about a point; for a point mass on a circle, $mr^2\omega$.
- **Lagrangian**: Kinetic energy minus potential energy, written in **generalised coordinates**.

Check yourself

- A 3 kg mass on a spring oscillates at 2.0 rad/s. What is k?
- Near x_0, $U''(x_0) = 50\ \mathrm{J/m^2}$ for a 2 kg particle. What is ω?
- A skater pulls her arms in and halves her moment of inertia. What happens to her angular speed and her kinetic energy?