Course Companion

GRE Subject Test · GRE Physics

Every topic handout in one volume

全部专题讲义合集

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Classical mechanics

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In 1666 Isaac Newton compared the falling apple with the Moon. The Moon is about 60 Earth radii away ($r = 3.84 \times 10^8$ m) and goes round once in 27.3 days, so its speed is about 1020 m/s. Its **centripetal acceleration** 向心加速度 is then $v^2/r \approx 2.7 \times 10^{-3}$ m/s^2, which is $g/60^2$. One inverse-square law explained both the apple and the Moon: the Moon is always falling, but its velocity keeps turning.

GRE Physics mechanics questions (about 20% of the test) check the same habits:

- Treat Newton's second law as a vector law, with a diagram and sensible axes.
- Pick the conservation law whose condition actually holds.
- Find small-oscillation frequencies from the shape of the potential.
- Use the Lagrangian or Hamiltonian when constraints make forces awkward.

Newton's second law in components

Write $\sum \mathbf{F} = m\mathbf{a}$ once for each axis. Draw a **free-body diagram** 受力分析图 for one object, then choose axes along and across the constraint so that one component of the acceleration is zero.

- Block on a frictionless incline of angle θ: along the slope $a = g\sin\theta$; across it, $N = mg\cos\theta$.
- Uniform circular motion: the net force points to the centre and has size mv^2/r. It is not an extra force; it is the sum of the real ones.
- Constant speed does not mean zero acceleration. A turning velocity needs a net force.

Choosing a conservation law

Each conservation law has its own condition. Check the condition, not the wording of the question.

- **Momentum** is conserved when the net external impulse is zero, for example during a short collision.
- **Mechanical energy** is conserved when no non-conservative force (friction, a sticking collision) does work.
- **Angular momentum** 角动量 about a point is conserved when the net external torque about that point is zero.

A collision can conserve momentum and still lose kinetic energy. If 2 kg at 3 m/s hits 1 kg at rest and they stick, momentum gives $6 = 3v$, so $v = 2$ m/s. Kinetic energy falls from 9 J to 6 J; 3 J becomes heat and deformation.

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Central forces and orbits

A central force points along the line to a fixed centre, so it exerts no torque about that centre. Angular momentum $L = mr^2\omega$ is constant, and ω falls as $1/r^2$.

- With $L/m = r^2\omega = 9$ m^2/s, the angular speed is 4 rad/s at 1.5 m and 1 rad/s at 3 m.
- This is Kepler's second law: equal areas in equal times.
- Radial motion can be studied with an effective potential, which adds a centrifugal term to $U(r)$:

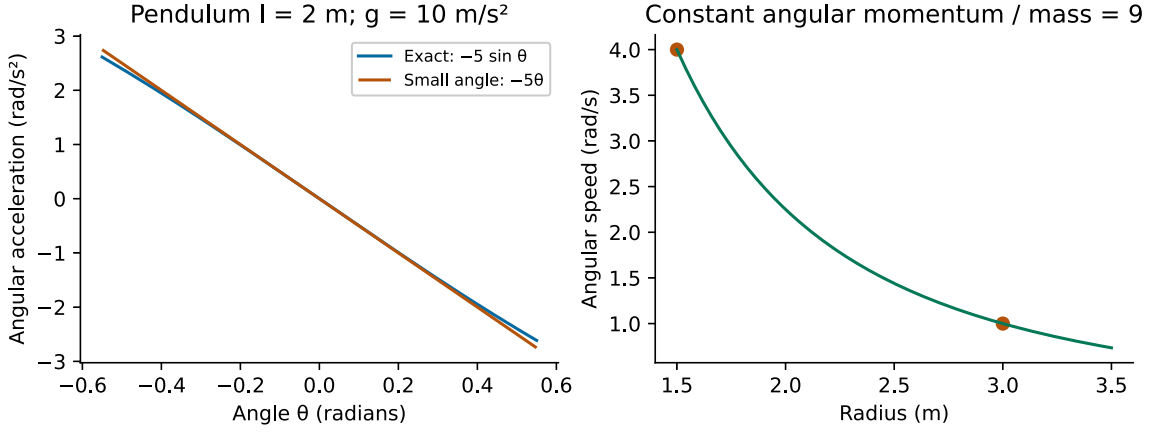
$$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2mr^2}$$

Small oscillations

Near a stable minimum x_0 of a potential $U(x)$, keep the first term that does not vanish. The restoring force is then proportional to the displacement, and the motion is simple harmonic.

$$U(x) \approx U(x_0) + \tfrac{1}{2}U''(x_0)(x - x_0)^2, \qquad \omega = \sqrt{\frac{U''(x_0)}{m}}$$

- Spring: $\omega = \sqrt{k/m}$. Doubling the mass divides ω by $\sqrt{2}$, not by 2.
- Simple pendulum: for small angles $\sin\theta \approx \theta$, so $\ddot{\theta} = -(g/\ell)\theta$ and $\omega = \sqrt{g/\ell}$. A 2 m pendulum has $\ddot{\theta} \approx -4.9\,\theta$.
- A minimum is stable only if $U''(x_0) > 0$. At a maximum, a small push grows instead of oscillating.



Small angles make the pendulum linear ($\ddot{\theta} \approx -5\theta$); constant angular momentum makes ω fall as $1/r^2$

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Lagrangian and Hamiltonian mechanics

The **Lagrangian** 拉格朗日量 is $L = T - V$, written in **generalised coordinates** 广义坐标 that already respect the constraints. Each coordinate q obeys the Euler–Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- If L does not contain a coordinate q (a cyclic coordinate), its momentum $p = \partial L/\partial \dot{q}$ is conserved.
- The **Hamiltonian** 哈密顿量 $H = \sum p\dot{q} - L$ equals $T + V$ when the constraints do not depend on time. Hamilton's equations are $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$.
- Constraint forces (a normal force, a string tension) never appear, which is why this method suits beads, rails and pendulums.

Worked example: a spring by Lagrange

A mass $m = 2.0$ kg on a spring of stiffness $k = 18$ N/m moves along x, measured from the equilibrium position.

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 \quad \Rightarrow \quad m\ddot{x} + kx = 0$$

- Angular frequency: $\omega = \sqrt{k/m} = \sqrt{18/2.0} = 3.0$ rad/s.
- Period: $T = 2\pi/\omega = 2\pi/3.0 = 2.09$ s.
- Check: k/m has units of s^{-2}, and positive stiffness gives oscillation, not runaway motion.

Worked example: rolling down a slope

A solid cylinder ($I = \frac{1}{2}mR^2$) rolls without slipping down a slope of 30°, starting from rest. Find its acceleration and its speed after it has dropped 1.2 m.

- Energy: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ with $\omega = v/R$, so $mgh = \frac{3}{4}mv^2$ and $v = \sqrt{4gh/3} = \sqrt{4 \times 9.8 \times 1.2/3} = 3.96$ m/s.
- Acceleration: $a = \dfrac{g\sin\theta}{1 + I/(mR^2)} = \frac{2}{3}g\sin 30° = 3.27$ m/s^2.
- Static friction makes it roll but does no work, because the contact point does not slide. So energy conservation still holds.

Common errors

- Adding "centripetal force" as an extra force on the diagram. It is the net inward force.
- Using kinetic-energy conservation in a collision just because momentum is conserved.

- Forgetting that angular momentum depends on the point chosen; choose the point where the torque is zero.
- Using $\omega = \sqrt{g/\ell}$ for large swings. It holds only while $\sin\theta \approx \theta$.
- Writing $L = T + V$. The Lagrangian is $T - V$; the Hamiltonian is usually $T + V$.

Key terms

centripetal acceleration: The acceleration toward the centre of a circular path, v^2/r.

free-body diagram: A sketch of one object showing every force acting on it.

angular momentum: The moment of linear momentum about a point; for a point mass on a circle, $mr^2\omega$.

Lagrangian: Kinetic energy minus potential energy, written in generalised coordinates.

generalised coordinates: Any independent variables that fix a system's configuration.

Hamiltonian: The energy function of a system in terms of coordinates and momenta.

Check yourself

- A 3 kg mass on a spring oscillates at 2.0 rad/s. What is k?
- Near x_0, $U''(x_0) = 50$ J/m^2 for a 2 kg particle. What is ω?
- A skater pulls her arms in and halves her moment of inertia. What happens to her angular speed and her kinetic energy?

Answers: $k = m\omega^2 = 12$ N/m. $\omega = \sqrt{50/2} = 5$ rad/s. Angular momentum is conserved, so ω doubles; $KE = L^2/(2I)$ doubles, the extra energy coming from the work her arms do.

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Electromagnetism

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In 1836 Michael Faraday built a room about 3.7 m on each side, covered it in metal foil and copper wire, and sat inside with sensitive electroscopes while a large electrostatic machine threw sparks at the outside. His instruments showed no charge and no field. The charge stayed on the outer surface of the **conductor** 导体, and the field inside was zero. Today the same idea protects people in cars and aircraft from lightning.

GRE Physics electromagnetism (about 18% of the test) asks you to:

- Get the field from the potential, and know that a value of V alone says nothing about $\mathbf{E}$.
- Use **Gauss's law** 高斯定律 only where symmetry makes it work.
- Solve circuits with Kirchhoff's rules and RC decay.
- Apply Faraday's law, Lenz's law and the properties of electromagnetic waves.

Field and potential

The **electric potential** 电势 V is energy per unit charge. The field is the downhill slope of the potential: $\mathbf{E} = -\nabla V$. So the field depends on how V changes, not on its value.

- Midway between $+q$ and $-q$, $V = 0$, but the field is strong and points from $+q$ to $-q$.
- Midway between two equal charges $+q$, $E = 0$, but $V \neq 0$.
- Point charge: $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ and $V = \dfrac{q}{4\pi\varepsilon_0 r}$. A charge q moved through ΔV gains energy $q\,\Delta V$.

Gauss's law needs symmetry

Gauss's law, $\oint \mathbf{E}\cdot d\mathbf{A} = Q_{\text{enc}}/\varepsilon_0$, is always true. It gives the total **flux** 通量 through a closed surface. It gives the field at each point only when symmetry makes E constant over the surface.

- Sphere (outside): $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$. Long line: $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$. Large plane: $E = \dfrac{\sigma}{2\varepsilon_0}$.
- Inside a uniformly charged shell, $E = 0$, and V is constant at its surface value, not zero.
- Without symmetry (a cube with a charge off-centre), Gauss's law still gives the total flux, but not E.

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Conductors in equilibrium

In a conductor at rest, free charges move until the field inside is zero. That one fact explains Faraday's room.

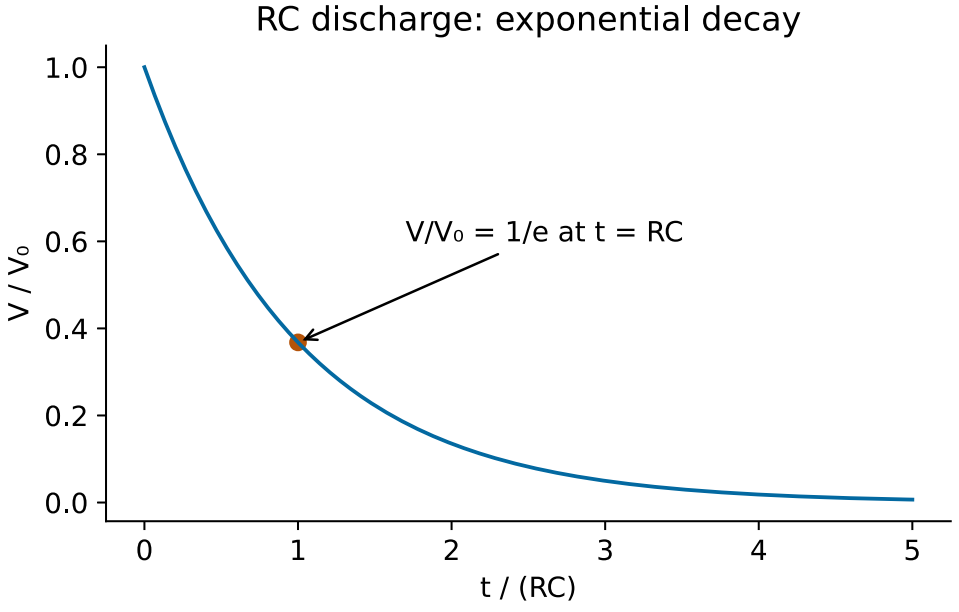
- Any excess charge sits on the outer surface; the whole conductor is at one potential.
- Just outside the surface, the field is perpendicular to it with size σ/ε_0.
- A closed metal shell screens its inside from outside charges. This is a Faraday cage.

Circuits and RC decay

Kirchhoff's rules are conservation laws: the currents into a node sum to zero (charge), and the voltage changes round any loop sum to zero (energy). A capacitor discharging through a resistor decays exponentially, with **time constant** 时间常数 $\tau = RC$.

$$V(t) = V_0\, e^{-t/RC}$$

- After one time constant, $V = V_0/e \approx 0.37\,V_0$; after 5τ, less than 1% is left.
- The time to fall to half is $\tau \ln 2 \approx 0.69\,\tau$.
- Energy stored in a capacitor: $\tfrac{1}{2}CV^2$. Halving the voltage leaves a quarter of the energy.



After one time constant RC the voltage has fallen to 1/e ≈ 0.37 of its start

Induction and electromagnetic waves

A changing magnetic flux Φ_B through a circuit drives an **electromotive force** 电动势: $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$. By **Lenz's law** 楞次定律, the induced current opposes the change in flux.

- A rod of length L moving at v across a field B has $\mathcal{E} = BLv$.
- In vacuum, Maxwell's equations give waves travelling at $c = 1/\sqrt{\mu_0\varepsilon_0}$, with $E = cB$ and $\mathbf{E} \perp \mathbf{B} \perp$ the direction of travel.
- The energy flow is $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$; the average intensity is $E_0^2/(2\mu_0 c)$.

Worked example: an RC discharge

A 10 μF capacitor charged to 12 V discharges through 100 kΩ.

- Time constant: $\tau = RC = (1.0 \times 10^5)(1.0 \times 10^{-5}) = 1.0$ s.
- After 2.0 s: $V = 12\,e^{-2} = 1.6$ V.
- Time to reach 3.0 V: $3 = 12\,e^{-t/1.0}$, so $t = \ln 4 = 1.39$ s.

Worked example: a line of charge

A long wire carries $\lambda = 2.0$ nC/m. Find E at 0.10 m.

- Symmetry: the field is radial and has the same size on a coaxial cylinder, so Gauss's law applies.
- Gauss: $E(2\pi r L) = \lambda L/\varepsilon_0$, so $E = \dfrac{\lambda}{2\pi\varepsilon_0 r} = \dfrac{2.0 \times 10^{-9}}{2\pi(8.85 \times 10^{-12})(0.10)} = 360$ N/C.
- Check: doubling r halves E for a line, but quarters it for a point charge.

Common errors

- Concluding $E = 0$ where $V = 0$, or $V = 0$ where $E = 0$.
- Using Gauss's law to find E for a shape without symmetry.
- Taking the potential inside a charged shell as zero; it equals the surface value.
- Getting the direction of an induced current from the field instead of from the change in flux.
- Forgetting that energy in a capacitor goes as V^2.

Key terms

conductor: A material with charges that move freely.

Gauss's law: The flux of **E** through a closed surface equals the enclosed charge divided by ε_0.

electric potential: Electric potential energy per unit charge.

flux: The amount of field passing through a surface.

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time constant: The time for an exponential decay to fall to $1/e$ of its start; RC for a capacitor and resistor.

electromotive force: The energy given to each unit of charge by a source, such as a changing flux.

Lenz's law: An induced current flows in the direction that opposes the change producing it.

Check yourself

- What is the field 0.50 m from a 1.0 nC point charge?
- A capacitor discharges through $R = 1.0$ MΩ with $C = 2.0$ μF. How long until the voltage has halved?
- A light wave has $E_0 = 300$ V/m. What is B_0?

Answers: $E = (9.0 \times 10^9)(1.0 \times 10^{-9})/0.25 = 36$ N/C. $\tau = 2.0$ s, so $t = 2.0 \ln 2 = 1.4$ s. $B_0 = E_0/c = 1.0 \times 10^{-6}$ T.

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Waves and optics

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In 1803 Thomas Young described a simple experiment to the Royal Society. He let a narrow beam of sunlight fall on a slip of card about 1/30 inch thick, so that light passed on both sides of it. In the shadow of the card he saw bright and dark bands. When he blocked the light on one side, the bands vanished. Light from two paths was adding and cancelling: it behaved as a wave.

GRE Physics optics and waves (about 9% of the test) asks you to:

- Add **coherent** 相干的 waves by amplitude, then square for intensity.
- Turn a **path difference** 程差 into a phase difference.
- Use the double-slit, single-slit and thin-film conditions.
- Decide when ray optics is good enough.

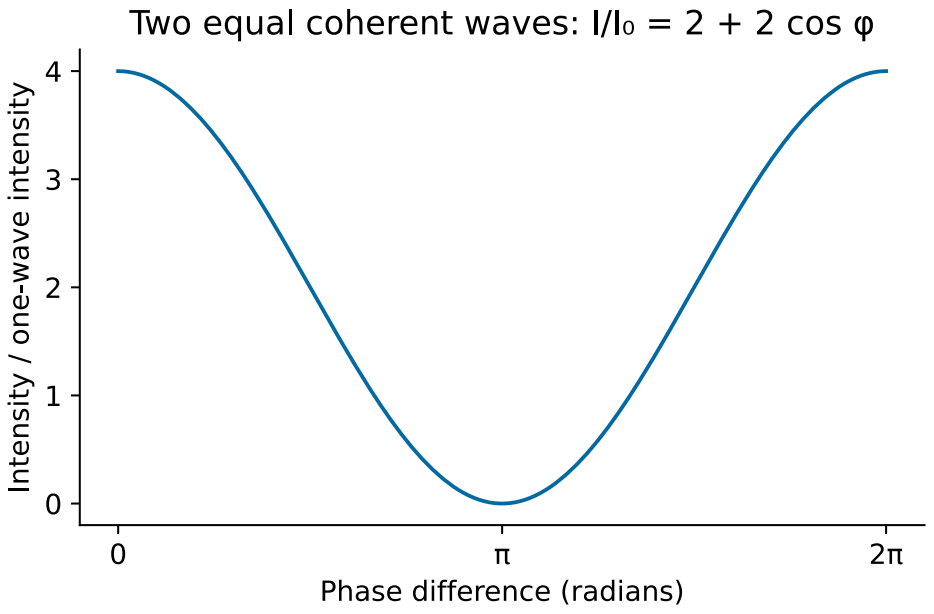
Superposition: add amplitudes, then square

When two waves meet, their displacements add (**superposition** 叠加). Intensity is proportional to the square of the amplitude, so you must add amplitudes first.

$$I = 2I_0(1 + \cos\varphi)$$

- Two equal coherent waves in phase: amplitude $2A$, intensity $4I_0$. In antiphase ($\varphi = \pi$): zero.
- Adding intensities would always give $2I_0$. That is right only for incoherent light, such as two separate lamps.
- Coherent sources keep a fixed phase relation; in practice they come from one source split in two.

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$I = 2I_0(1 + \cos\varphi)$: four times one wave in phase, zero in antiphase

Path difference and the double slit

A path difference Δ gives a phase difference $\varphi = 2\pi\Delta/\lambda$. Bright **fringes** 条纹 appear where the path difference is a whole number of wavelengths.

- Double-slit maxima: $d\sin\theta = m\lambda$, $m = 0, \pm1, \pm2, \ldots$
- On a distant screen, the fringe spacing is $\Delta y = \lambda L/d$. Narrower slit spacing gives wider fringes.
- With N slits (a grating), the maxima stay at $d\sin\theta = m\lambda$ but become much sharper.

Single-slit diffraction

A single slit of width a spreads light by **diffraction** 衍射. Waves from different parts of the slit cancel at the dark minima.

- Minima at $a\sin\theta = m\lambda$ with $m = \pm1, \pm2, \ldots$ Note: $m = 0$ is the bright centre, not a minimum.
- The central maximum is twice as wide as the others: $2\lambda L/a$ on a screen.
- A narrower slit spreads the light more. In a double slit, this single-slit pattern is the envelope of the fringes.

Thin films

Light reflected from the top and bottom of a **thin film** 薄膜 interferes. A reflection off a medium of higher refractive index adds a half-wavelength phase shift.

- Soap film in air: one reflection is shifted, so strong reflection needs $2nt = (m+\tfrac{1}{2})\lambda$.
- If both reflections are shifted (or neither), strong reflection needs $2nt = m\lambda$.
- Inside the film the wavelength is λ/n.

Rays or waves?

Ray optics works when apertures and objects are much larger than the wavelength; wave effects dominate when they are comparable. For a thin lens:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}, \qquad m = -\frac{d_i}{d_o}$$

- With this sign convention, a real image has $d_i > 0$ and is inverted.
- A converging lens with the object inside f gives a virtual, upright, magnified image.

Worked example: slits and screens

Light of 600 nm falls on a double slit with $d = 0.30$ mm; the screen is 2.0 m away.

- Fringe spacing: $\Delta y = \dfrac{\lambda L}{d} = \dfrac{(6.0\times10^{-7})(2.0)}{3.0\times10^{-4}} = 4.0$ mm.
- One slit alone, width $a = 0.10$ mm: central maximum width $2\lambda L/a = 2(6.0\times10^{-7})(2.0)/(1.0\times10^{-4}) = 24$ mm.
- So about six fringes fit inside the central diffraction peak.

Worked example: a soap film

Find the thinnest soap film ($n = 1.33$) that strongly reflects green light of 532 nm.

- Only the top reflection (air to soap) is shifted, so use $2nt = (m+\tfrac{1}{2})\lambda$ with $m = 0$.
- $t = \dfrac{\lambda}{4n} = \dfrac{532}{4\times1.33} = 100$ nm.
- A film much thinner than this reflects almost nothing, so it looks black.

Common errors

- Adding intensities for coherent waves.
- Using $m = 0$ as a single-slit minimum.
- Forgetting the half-wavelength shift on reflection from a denser medium.
- Using the vacuum wavelength inside a film instead of λ/n.
- Mixing sign conventions in the lens equation.

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Key terms

coherent: Having a fixed phase relationship over time.

path difference: The difference in the distances two waves travel to the same point.

superposition: Adding the displacements of waves that meet.

fringe: One of the bright or dark bands in an interference pattern.

diffraction: The spreading of a wave through an opening or round an edge.

thin film: A layer about as thick as a wavelength, whose two reflections interfere.

Check yourself

- Two coherent waves of intensity I_0 meet with a path difference of $\lambda/4$. What is the intensity?
- The slit spacing of a double slit is halved. What happens to the fringe spacing?
- An object is 30 cm from a lens with $f = 10$ cm. Where is the image?

Answers: $\varphi = \pi/2$, so $I = 2I_0(1 + 0) = 2I_0$. It doubles. $1/d_i = 1/10 - 1/30$, so $d_i = 15$ cm: a real, inverted image.

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Thermodynamics and statistical mechanics

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In 1824 the French engineer Sadi Carnot published *Reflections on the Motive Power of Fire*. Steam engines then turned only a few per cent of their heat into work, and inventors argued about better fluids and designs. Carnot showed that the best possible engine depends only on the temperatures of its hot and cold **reservoirs** 热库. Today we write his limit as $\eta \le 1 - T_c/T_h$, a consequence of **entropy** 熵.

GRE Physics thermodynamics and statistical mechanics (about 10% of the test) asks you to:

- Use the **first law** 热力学第一定律 with a stated sign convention.
- Apply Carnot limits and entropy changes.
- Weight states with Boltzmann factors and a partition function.
- Choose Fermi–Dirac, Bose–Einstein or Maxwell–Boltzmann statistics.

The first law and its signs

Energy is conserved: $\Delta U = Q - W$, where Q is heat added to the system and W is work done by the system. Some texts use work done on the system instead; always say which.

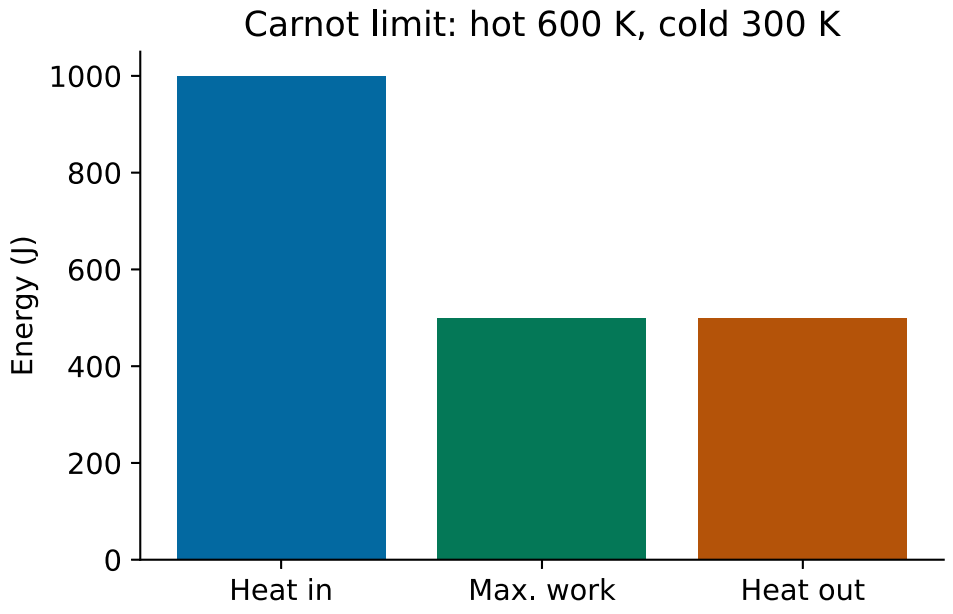
- Isothermal ideal gas: $\Delta U = 0$, so $Q = W = nRT\ln(V_f/V_i)$.
- Isochoric (constant volume): $W = 0$, so $Q = \Delta U = nC_V\Delta T$.
- **Adiabatic** 绝热的 ($Q = 0$): PV^γ is constant, and the gas cools as it expands.
- On a P–V diagram, the work is the area under the path.

Carnot limits

No engine working between T_h and T_c beats a reversible one. Temperatures must be in kelvin.

$$\eta \le 1 - \frac{T_c}{T_h}, \qquad \mathrm{COP}_{\text{fridge}} \le \frac{T_c}{T_h - T_c}$$

- Between 600 K and 300 K, at most half of the heat input becomes work.
- A refrigerator works better when the two temperatures are close.



Carnot efficiency $1 - 300/600 = 1/2$ caps the work at half the heat input

Entropy

For reversible heat flow at temperature T, $\Delta S = Q/T$. The second law says the total entropy of an isolated system never decreases.

- Entropy is a state function. For an irreversible process, find ΔS along any reversible path between the same states.
- Free expansion of an ideal gas to double its volume: $W = 0$ and $Q = 0$, yet $\Delta S = nR \ln 2 > 0$.
- Statistically, $S = k \ln \Omega$, where Ω is the number of microstates.

Boltzmann factors

In equilibrium at temperature T, a state of energy E_i has a probability proportional to its **Boltzmann factor** 玻尔兹曼因子 $e^{-E_i/kT}$.

$$P(E_i) = \frac{g_i\, e^{-E_i/kT}}{Z}, \qquad Z = \sum_i g_i\, e^{-E_i/kT}$$

- Z is the partition function 配分函数; g_i counts states with the same energy.
- Two levels a gap ΔE apart: $N_2/N_1 = (g_2/g_1)\, e^{-\Delta E/kT}$.
- At room temperature, $kT \approx 0.025$ eV.

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Which statistics?

Identical quantum particles fill states differently. The mean number in a state of energy E is:

$$\bar{n} = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$

- Plus sign: **fermions** 费米子 (electrons, protons), at most one per state.
- Minus sign: **bosons** 玻色子 (photons, helium-4 atoms), which can pile into one state.
- When $\bar{n} \ll 1$ (a hot, dilute gas), both reduce to Maxwell–Boltzmann.

Worked example: an impossible engine

An inventor claims an engine that turns 70% of its heat into work, running between 600 K and 300 K.

- Carnot limit: $\eta_{\max} = 1 - 300/600 = 0.50$. The claim breaks the second law, though not the first.
- Best case, per 1000 J taken in: 500 J of work and 500 J rejected.
- Entropy check: $\Delta S_{\text{hot}} = -1000/600 = -1.67$ J/K and $\Delta S_{\text{cold}} = +500/300 = +1.67$ J/K. The total is zero, as a reversible engine needs. The claimed engine would reject only 300 J, giving a total of -0.67 J/K: impossible.

Worked example: expansion and level populations

One mole of ideal gas expands isothermally at 300 K to twice its volume.

- Work: $W = nRT \ln 2 = (1)(8.31)(300)(0.693) = 1.73$ kJ, and $Q = W$ because $\Delta U = 0$.
- Entropy: $\Delta S = Q/T = 1730/300 = 5.76$ J/K, the same as $R \ln 2$.
- Two levels 0.10 eV apart at 300 K ($kT = 0.0259$ eV): $N_2/N_1 = e^{-3.87} = 0.021$, so about 2% are excited.

Common errors

- Using degrees Celsius in $1 - T_c/T_h$.
- Mixing the two sign conventions for work.
- Saying free expansion has no entropy change because $Q = 0$.
- Forgetting the degeneracy g_i in a Boltzmann ratio.
- Putting two electrons in the same quantum state.

Key terms

reservoir: A body large enough that heat flow does not change its temperature.

entropy: A state function that measures the number of ways a state can be arranged; $\Delta S = Q/T$ for reversible heat flow.

first law: Energy conservation for heat and work: $\Delta U = Q - W$.

adiabatic: With no heat exchanged.

Boltzmann factor: The weight $e^{-E/kT}$ of a state of energy E in equilibrium.

partition function: The sum of all Boltzmann weights, which normalises the probabilities.

fermion: A particle with half-integer spin; at most one per quantum state.

boson: A particle with integer spin; any number can share a state.

Check yourself

- What is the best possible COP of a refrigerator between 270 K and 300 K?
- Two moles of ideal gas expand freely into a vacuum, doubling their volume. What is ΔS?
- Which statistics describe the conduction electrons in a metal?

Answers: $270/30 = 9$. $\Delta S = 2R\ln 2 = 11.5$ J/K. Fermi–Dirac, because electrons are fermions and the electron gas is far from dilute.

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Quantum mechanics and atomic physics

GRE Subject Test · GRE Physics

Quantum mechanics

In 1927 Clinton Davisson and Lester Germer fired a beam of 54 eV electrons at a nickel crystal. Most scattered weakly in all directions, but at 50° they found a strong beam, just like X-rays reflecting from crystal planes. The electrons were diffracting. Louis de Broglie's rule $\lambda = h/p$ gives 0.167 nm for these electrons, which matched the spacing of the nickel atoms. Matter has wave properties, described by a **wavefunction** 波函数.

GRE Physics quantum mechanics (about 12% of the test) asks you to:

- Normalise wavefunctions and read probabilities from coefficients.
- Use operators, eigenvalues and expectation values.
- Know the infinite well and harmonic oscillator by heart.
- Decide which observables commute.

Wavefunctions and normalisation

The probability of finding a particle between x and $x+dx$ is $|\psi(x)|^2 dx$. **Normalisation** 归一化 means the total probability is one: $\int |\psi|^2 dx = 1$.

- Normalise first, then calculate any probability.
- ψ must be continuous; where V is finite, $d\psi/dx$ is continuous too.
- A wavefunction multiplied by a constant phase $e^{i\alpha}$ describes the same state.

Measurement probabilities

Write the state in the orthonormal eigenstates ϕ_j of the quantity measured: $\psi = \sum_j c_j \phi_j$. Then $\sum_j |c_j|^2 = 1$.

- A measurement gives one **eigenvalue** 本征值 E_j, with probability $|c_j|^2$.
- After the measurement, the state is ϕ_j; measuring again at once gives E_j again.
- Relative phases between the c_j change position densities, but not these energy probabilities.

Observables and expectation values

Each observable is a **Hermitian** 厄米的 operator, so its eigenvalues are real. The **expectation value** 期望值 is the average over many identical preparations:

$$\langle A\rangle = \int \psi^* \hat{A}\,\psi\,dx = \sum_j |c_j|^2 a_j$$

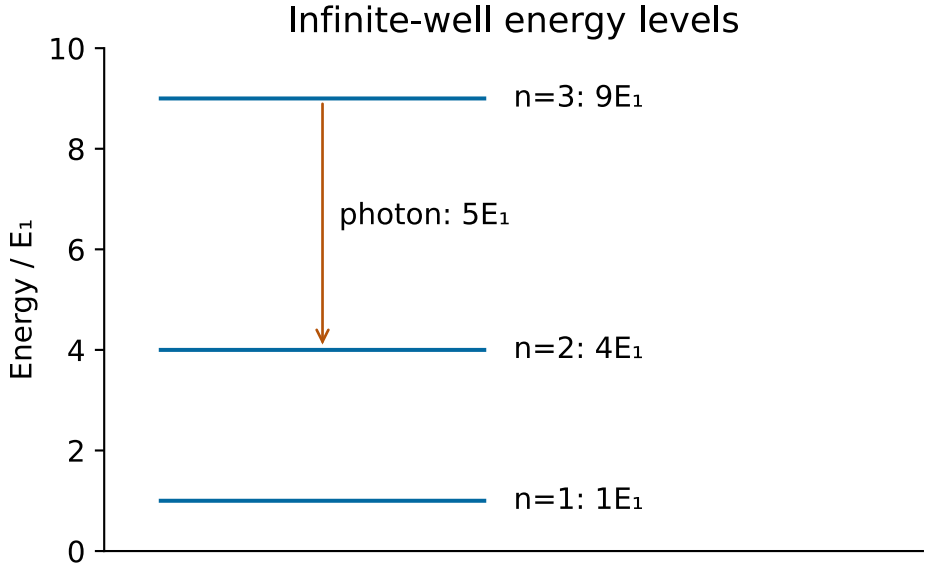
- $\langle A\rangle$ need not be a possible single result.
- Momentum: $\hat{p} = -i\hbar\, d/dx$. Energy: $\hat{H} = \hat{p}^2/2m + V(x)$.

The infinite well

A particle in $0 < x < L$ with infinite walls has standing waves that vanish at both walls.

$$\psi_n = \sqrt{\tfrac{2}{L}} \sin\frac{n\pi x}{L}, \qquad E_n = \frac{n^2 h^2}{8mL^2} = n^2 E_1, \quad n = 1, 2, 3, \ldots$$

- The lowest energy is not zero: a confined particle always moves. $n = 0$ would give $\psi = 0$, which is not a state.
- Doubling L divides every level by 4.



Levels scale as n^2, so the $3 \to 2$ photon carries $9E_1 - 4E_1 = 5E_1$

The harmonic oscillator

For $V = \frac{1}{2}m\omega^2 x^2$ the levels are equally spaced:

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

- The ground state has **zero-point energy** 零点能 $\frac{1}{2}\hbar\omega$ and a Gaussian wavefunction.

- Its photons all have energy $\hbar\omega$, because neighbouring levels are $\hbar\omega$ apart.

Commutators and uncertainty

The **commutator** 对易子 of two operators is $[A, B] = AB - BA$. Observables that commute share eigenstates and can have definite values together.

- $[x, p] = i\hbar$, which leads to $\Delta x\,\Delta p \geq \hbar/2$.
- p commutes with $p^2/2m$ but not with $V(x)$, so momentum is conserved only for a free particle.
- $[L_x, L_y] = i\hbar L_z$, but L^2 commutes with each component: you can know L^2 and L_z together.

Worked example: a superposition

A particle in an infinite well is in $\psi = (\phi_1 + 2\phi_2)/\sqrt{5}$.

- Check normalisation: $(1^2 + 2^2)/5 = 1$.
- Probabilities: $P(E_1) = 1/5$ and $P(E_2) = 4/5$.
- Average energy: $\langle E\rangle = \frac{1}{5}E_1 + \frac{4}{5}(4E_1) = 3.4\,E_1$, which is not one of the allowed energies.

Worked example: an electron in a 1 nm box

An electron ($m = 9.11 \times 10^{-31}$ kg) is confined to $L = 1.0$ nm.

- Ground state: $E_1 = \dfrac{h^2}{8mL^2} = \dfrac{(6.63\times10^{-34})^2}{8(9.11\times10^{-31})(1.0\times10^{-18})} = 6.0\times10^{-20}$ J $= 0.38$ eV.
- The $3 \to 2$ photon has $5E_1 = 1.9$ eV, so $\lambda = 1240/1.9 = 660$ nm: red light.
- This is roughly why some dye molecules with long chains are coloured.

Common errors

- Calculating probabilities before normalising.
- Treating $\langle E\rangle$ as a possible measured result.
- Starting the infinite well at $n = 0$, or the oscillator at $n = 1$.
- Mixing h and $\hbar$: $n^2h^2/8mL^2$ and $n^2\pi^2\hbar^2/2mL^2$ are the same, but mixed forms are off by $4\pi^2$.
- Assuming L_x and L_z can both be definite.

Key terms

wavefunction: The function whose squared size gives the probability density of a particle.

normalisation: Scaling a wavefunction so the total probability is one.

eigenvalue: A possible result of measuring an observable.

Hermitian: Equal to its own conjugate transpose, so with real eigenvalues.

expectation value: The average result over many identical preparations.

zero-point energy: The lowest energy a confined quantum system can have, which is above the minimum of the potential.

commutator: The operator $AB - BA$.

Check yourself

- The width of an infinite well is doubled. What happens to E_1?
- A harmonic oscillator has $\hbar\omega = 0.20$ eV. What are its ground-state energy and the energy of its photons?
- Can a measurement give definite values of both L_x and L_z?

Answers: E_1 falls to a quarter. 0.10 eV, and 0.20 eV. No: $[L_x, L_z] \neq 0$, so they have no common eigenstates (except when $L = 0$).

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Relativity, laboratory methods and specialised topics

GRE Subject Test · GRE Physics

Relativity and laboratory methods

In October 1971 Joseph Hafele and Richard Keating flew four caesium atomic clocks round the world on ordinary airliners, first eastward and then westward. Compared with clocks left at the US Naval Observatory, the eastward clocks lost about 59 ns and the westward clocks gained about 273 ns. Relativity predicted -40 ± 23 ns and $+275 \pm 21$ ns. The experiment tested **time dilation** 时间膨胀, and its result was only as good as its uncertainties.

GRE Physics special relativity (about 6%) and laboratory methods (about 6%) ask you to:

- Use the **Lorentz factor** 洛伦兹因子 for time, length, energy and momentum.
- Remember that kinetic energy is not $\tfrac{1}{2}mv^2$ at high speed.
- Combine uncertainties and tell **precision** 精密度 from **accuracy** 准确度.
- Read straight lines from log plots.

Time dilation and length contraction

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

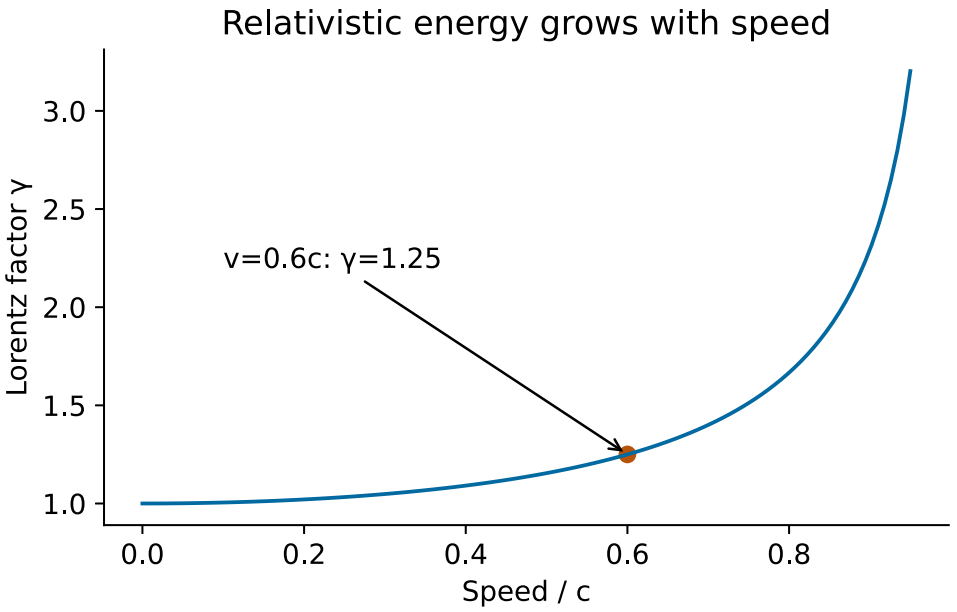
- A moving clock runs slow: $\Delta t = \gamma\,\Delta\tau$, where $\Delta\tau$ is the proper time in the clock's own frame.
- A moving rod is shorter along its motion: $L = L_0/\gamma$.
- Muons made high in the atmosphere live 2.2 μs at rest, yet many reach the ground because their lifetime is dilated in our frame.

Relativistic energy and momentum

Every particle has a **rest energy** 静能 mc^2. At speed v:

$$E = \gamma mc^2, \qquad p = \gamma mv, \qquad E^2 = (pc)^2 + (mc^2)^2$$

- Kinetic energy is $K = (\gamma - 1)mc^2$. At $v = 0.6c$, $\tfrac{1}{2}mv^2 = 0.18\,mc^2$, but $K = 0.25\,mc^2$.
- For a photon, $m = 0$ and $E = pc$.
- Mass is not "relativistic mass"; m is the same in every frame.



At $v = 0.6c$, $\gamma = 1/\sqrt{(1 - 0.36)} = 1.25$

Combining uncertainties

Independent uncertainties add in **quadrature** 方和根: the square root of the sum of squares.

- For a sum or difference, add the absolute uncertainties in quadrature.
- For a product or quotient, add the relative (percentage) uncertainties in quadrature: 3% and 4% give 5%.
- For $y = x^n$, multiply the relative uncertainty by $|n|$. Simple addition gives a worst case, not the standard uncertainty.

Precision, accuracy and counting

Random scatter limits precision; a calibration bias, a **systematic error** 系统误差, limits accuracy.

- Averaging N readings shrinks the random error by $1/\sqrt{N}$ but leaves a bias unchanged.
- When counting random events (radioactive decays, photons), N counts have an uncertainty of $\sqrt{N}$, a relative uncertainty of $1/\sqrt{N}$.

Log plots

Logarithms turn powers and exponentials into straight lines.

- If $y = Ax^n$, then $\log y = \log A + n \log x$: on log–log axes the line has slope n.
- If $y = Ae^{kx}$, then $\ln y = \ln A + kx$: a semi-log plot is straight with slope k.
- Read the slope from the logs of the values, not from the printed tick labels.

Worked example: an electron at 0.6c

An electron ($mc^2 = 0.511$ MeV) moves at $0.6c$.

- $\gamma = 1/\sqrt{1 - 0.36} = 1.25$, so $E = 0.639$ MeV and $K = 0.25 \times 0.511 = 0.128$ MeV.
- Momentum: $pc = \gamma mvc = 1.25 \times 0.6 \times 0.511 = 0.383$ MeV.
- Check: $E^2 = 0.408$ and $(pc)^2 + (mc^2)^2 = 0.147 + 0.261 = 0.408$ MeV^2.

Worked example: uncertainty in g

A pendulum gives $g = 4\pi^2 L/T^2$, with L known to 1.0% and T to 0.50%.

- The T^2 term doubles its relative uncertainty: $2 \times 0.50\% = 1.0\%$.
- In quadrature: $\sqrt{1.0^2 + 1.0^2} = 1.4\%$.
- A counter records 400 decays: 400 ± 20, or 5%. For 1%, it needs $1/\sqrt{N} = 0.01$, so 10,000 counts.

Common errors

- Using $\frac{1}{2}mv^2$ for kinetic energy at speeds near c.
- Applying time dilation the wrong way round: the proper time is the shortest.
- Adding uncertainties linearly when they are independent.
- Hoping that more readings will remove a systematic error.
- Reading a log–log slope from the tick labels instead of their logarithms.

Key terms

time dilation: A moving clock runs slow by the factor γ compared with clocks at rest.

Lorentz factor: $\gamma = 1/\sqrt{1 - v^2/c^2}$.

rest energy: The energy mc^2 a particle has when it is at rest.

precision: How closely repeated measurements agree.

accuracy: How close a measurement is to the true value.

quadrature: Combination as the square root of the sum of squares.

systematic error: An error that shifts every reading the same way.

Check yourself

- What is γ at $v = 0.8c$?
- Two independent relative uncertainties are 6% and 8%. What is the uncertainty of their product?
- A log–log plot of y against x is a straight line of slope -2. How does y depend on x?

Answers: $\gamma = 1/\sqrt{1 - 0.64} = 5/3$. $\sqrt{36 + 64} = 10\%$. $y \propto x^{-2}$, an inverse-square law.

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Atomic spectra and selection rules

GRE Subject Test · GRE Physics

In 1885 Johann Balmer, a Swiss schoolteacher, looked at the wavelengths of the four visible lines of hydrogen measured by Anders Ångström. He found that one formula fitted them all: $\lambda = B\,n^2/(n^2-4)$, with $B = 364.6$ nm and $n = 3, 4, 5, 6$. He had no theory for it. Twenty-eight years later Niels Bohr explained it with energy levels: each line is a jump from level n down to level 2.

GRE Physics atomic physics (about 10% of the test) asks you to:

- Find photon energies and wavelengths from hydrogen-like levels.
- Count states with the **quantum numbers** 量子数.
- Apply electric-dipole **selection rules** 选择定则.
- Know how fine structure, the Zeeman effect and shielding change the simple picture.

Hydrogen-like energies

For one electron round a nucleus of charge Ze:

$$E_n = -\frac{13.6\,Z^2}{n^2}\ \text{eV}, \qquad \lambda = \frac{hc}{E_\gamma}, \quad hc \approx 1240\ \text{eV nm}$$

- The photon energy is the positive difference $E_\gamma = E_i - E_f$.
- The **ionisation energy** 电离能 from the ground state is $13.6\,Z^2$ eV: 13.6 eV for H, 54.4 eV for He^+.
- Every level of He^+ ($Z = 2$) is four times deeper than the matching level of hydrogen.

Spectral series

A **spectral series** 谱线系 is the set of lines ending on one lower level.

- Lyman series: down to $n = 1$, in the ultraviolet ($2 \to 1$ is 10.2 eV, 122 nm).
- Balmer series: down to $n = 2$, mostly visible ($3 \to 2$ is the red line at 656 nm).
- Paschen series: down to $n = 3$, in the infrared.

Quantum numbers

Each electron state has n, $l = 0, \ldots, n-1$, $m_l = -l, \ldots, +l$ and spin $m_s = \pm\frac{1}{2}$.

- Level n holds $2n^2$ states: 2, 8, 18, …
- The total angular momentum is $\mathbf{J} = \mathbf{L} + \mathbf{S}$; for one electron with $l > 0$, $j = l \pm \frac{1}{2}$.
- Letters: $l = 0, 1, 2, 3$ are s, p, d, f.

Electric-dipole selection rules

A photon carries one unit of angular momentum, so a strong (electric-dipole) transition needs:

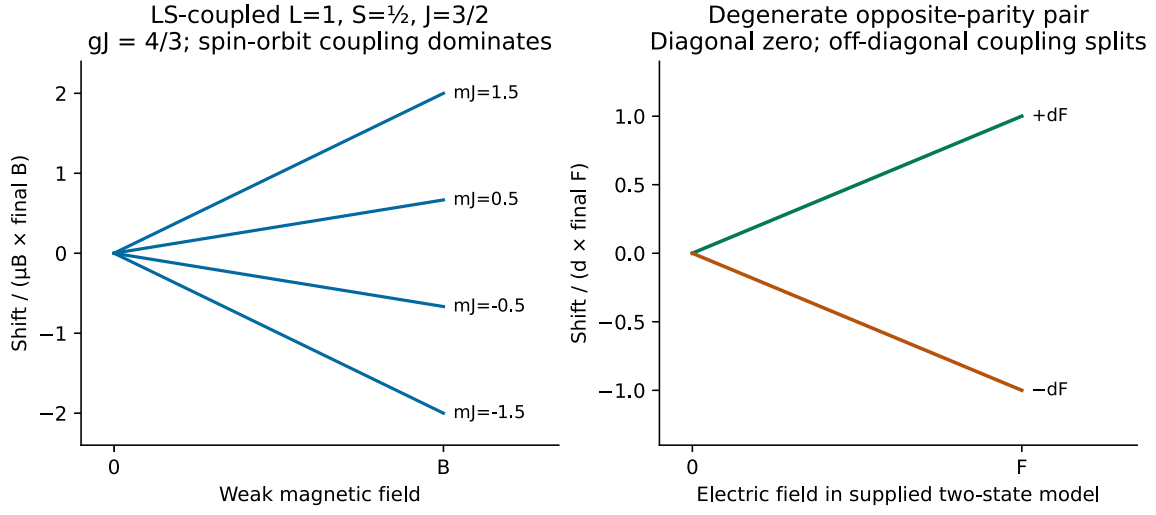
- $\Delta l = \pm 1$, $\Delta m_l = 0, \pm 1$, and no change of spin.
- $2p \to 1s$ ($\Delta l = -1$) is allowed and fast. $2s \to 1s$ ($\Delta l = 0$) is forbidden, so 2s is **metastable** 亚稳态的: it decays slowly, by emitting two photons.
- "Forbidden" means forbidden for this mechanism, not impossible.

Fine structure and the Zeeman effect

Spin–orbit coupling splits levels with the same l but different j: this is **fine structure** 精细结构. In a weak magnetic field B, the **Zeeman effect** 塞曼效应 splits each j level into $2j + 1$ sublevels:

$$\Delta E = g_J \, m_J \, \mu_B B, \qquad \mu_B = 5.79 \times 10^{-5} \text{ eV/T}$$

- For ${}^2P_{3/2}$ ($L = 1$, $S = 1/2$, $J = 3/2$), $g_J = 4/3$, and there are four equally spaced sublevels.
- An electric field splits a degenerate pair of opposite-parity states linearly in the field (the linear Stark effect of hydrogen).



Weak-field Zeeman shifts are gJ · mJ · μB · B; an electric field splits a degenerate opposite-parity pair linearly

Many-electron atoms

Inner electrons shield the nucleus, so an outer electron feels much less than the full charge Ze.

- Shielding breaks the l-degeneracy: in sodium, 3s lies below 3p. The 3p $\to$ 3s jump gives the yellow D line at 589 nm.
- You cannot put $Z = 11$ into $-13.6Z^2/n^2$ for sodium's outer electron.
- The Pauli principle fills states in order, giving the periodic table.

Worked example: Balmer's red line

Find the wavelength of the $3 \to 2$ line in hydrogen, and the He^+ line with the same photon energy as hydrogen's $2 \to 1$.

- Hydrogen: $E_\gamma = 13.6\left(\frac{1}{4} - \frac{1}{9}\right) = 1.89$ eV, so $\lambda = 1240/1.89 = 656$ nm.
- He^+ $4 \to 2$: $E_\gamma = 13.6 \times 4\left(\frac{1}{4} - \frac{1}{16}\right) = 10.2$ eV, the same as hydrogen's Lyman-alpha (122 nm).
- Check with Balmer's formula: $B \times 9/5 = 364.6 \times 1.8 = 656$ nm.

Worked example: Zeeman splitting

A $^2P_{3/2}$ level sits in a field of 1.0 T.

- Neighbouring sublevels: $g_J \mu_B B = \frac{4}{3}(5.79 \times 10^{-5})(1.0) = 7.7 \times 10^{-5}$ eV apart.
- Largest shift ($m_J = 3/2$): $\frac{4}{3} \times \frac{3}{2} \times 5.79 \times 10^{-5} = 1.16 \times 10^{-4}$ eV.
- These shifts are tiny next to the 2 eV photon energies of visible light, so the lines split only slightly.

Common errors

- Giving a negative photon energy, or using $E_f - E_i$.
- Forgetting the factor Z^2 for hydrogen-like ions.
- Calling a $\Delta l = 0$ or $\Delta l = 2$ transition allowed.
- Counting n^2 states per level instead of $2n^2$, which includes spin.
- Using the hydrogen formula for outer electrons of many-electron atoms.

Key terms

quantum numbers: The labels n, l, m_l, m_s (and j, m_J) that identify an electron state.

selection rules: Conditions on quantum-number changes that a given transition mechanism allows.

ionisation energy: The energy needed to remove an electron completely.

spectral series: All the lines that end on the same lower level.

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metastable: Long-lived because its decay route is forbidden at leading order.

fine structure: Small splittings of atomic levels from spin–orbit and relativistic effects.

Zeeman effect: The splitting of spectral lines in a magnetic field.

Check yourself

- What is the ionisation energy of ground-state He^{+}?
- Is the transition 3d $\rightarrow$ 2s allowed for electric-dipole emission?
- How many electron states does the $n = 3$ level hold?

Answers: 54.4 eV. No: $\Delta l = -2$. $2 \times 3^2 = 18$.

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Solid-state, nuclear and particle models

GRE Subject Test · GRE Physics

Solid-state, nuclear and particle physics

In December 1947 John Bardeen and Walter Brattain, at Bell Laboratories, pressed two gold contacts very close together onto a small slab of germanium. A weak signal fed into one contact came out amplified. This was the first transistor. It worked because of what the germanium contained: a tiny amount of impurity, a few atoms in a million, changed how many charge carriers it had. Electrons alone do not decide conduction; empty states near the top of the electron energies do.

GRE Physics solid-state, nuclear and particle questions (part of the specialised topics, about 9%) ask you to:

- Explain conduction with bands, the **band gap** 带隙 and **doping** 掺杂.
- Use Fermi–Dirac occupancy near the **Fermi level** 费米能级.
- Work with decay laws, **activity** 活度 and **half-life** 半衰期.
- Find **binding energy** 结合能 and check conservation laws in reactions.

Energy bands

In a solid, the energy levels of many atoms merge into bands. A band that is completely full carries no net current, because every electron moving one way is matched by one moving the other way.

- Metals have a partly filled band, so they conduct at any temperature.
- Insulators have a full band and a large gap (diamond: about 5.5 eV).
- Semiconductors have a small gap (silicon: 1.1 eV), so heat lifts a few electrons across it; their conductivity rises with temperature.

Doping

Adding impurity atoms changes the number of carriers without changing the band structure much.

- Donors (phosphorus or arsenic in silicon) have one extra electron, which easily enters the conduction band: n-type.
- Acceptors (boron or aluminium) take an electron and leave a hole in the valence band: p-type.
- One phosphorus atom per million silicon atoms gives about $5 \times 10^{22}\ \mathrm{m}^{-3}$ electrons, far more than pure silicon has at room temperature.

Fermi–Dirac occupancy

Electrons are fermions, so the mean occupation of a state of energy E is:

$$f(E) = \frac{1}{e^{(E-\mu)/kT} + 1}$$

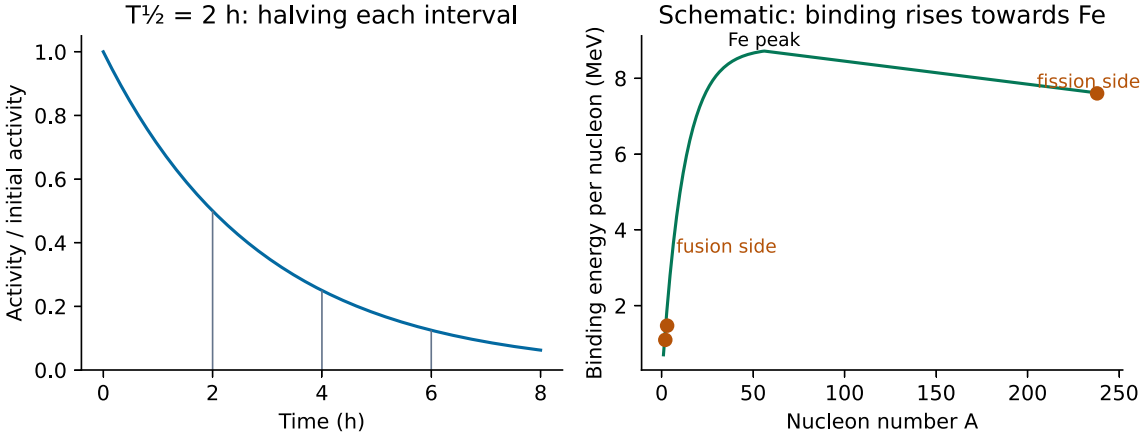
- At $T = 0$, f is a step: 1 below μ and 0 above.
- At $E = \mu$, $f = \frac{1}{2}$ at every temperature.
- Heating changes f only in a band about kT wide around μ, which is why most electrons in a metal cannot take up heat.

Radioactive decay

Each nucleus has the same chance of decaying per second, λ, so the number left falls exponentially.

$$N = N_0\, e^{-\lambda t}, \qquad A = \lambda N, \qquad t_{1/2} = \frac{\ln 2}{\lambda}$$

- After k half-lives, $N/N_0 = 2^{-k}$, and the activity falls by the same factor.
- Activity counts decays per second (becquerel, Bq); N counts nuclei.



Each half-life halves the activity; energy is released by moving towards the iron peak, by fusion from the left or fission from the right

Binding energy

A nucleus weighs less than its separate protons and neutrons. The missing mass Δm is the binding energy: $E_B = \Delta m\, c^2$, with $1\ \mathrm{u}\, c^2 = 931.5$ MeV.

- Stability follows the binding energy per nucleon, which peaks near iron at about 8.8 MeV.

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- Fusing light nuclei or splitting heavy ones moves towards the peak and releases energy.

Conservation laws in reactions

Every reaction or decay must conserve charge, energy, momentum, angular momentum, baryon number and each lepton number.

- Beta-minus decay is $n \to p + e^- + \bar{\nu}_e$. The antineutrino keeps lepton number at zero and shares the energy, which is why the electron energies form a continuous spectrum.
- Protons and neutrons have baryon number $+1$; electrons and neutrinos have electron-lepton number $+1$; antiparticles have the opposite sign.

Worked example: a decaying sample

A sample with a half-life of 2.0 h has an activity of 800 Bq.

- After 6.0 h (three half-lives): $A = 800/2^3 = 100$ Bq.
- Decay constant: $\lambda = \ln 2/(7200\ \text{s}) = 9.6 \times 10^{-5}\ \text{s}^{-1}$.
- Nuclei at the start: $N = A/\lambda = 800/(9.6 \times 10^{-5}) = 8.3 \times 10^6$.

Worked example: binding energy of helium-4

Two protons (1.007276 u each) and two neutrons (1.008665 u each) form a helium-4 nucleus of 4.001506 u.

- Separate particles: $2(1.007276) + 2(1.008665) = 4.031882$ u.
- Mass defect: 0.030376 u, so $E_B = 0.030376 \times 931.5 = 28.3$ MeV.
- Per nucleon: $28.3/4 = 7.07$ MeV, which is why helium-4 is so stable for a light nucleus.

Common errors

- Thinking more electrons always means better conduction; a full band carries no current.
- Saying $f(\mu) = 1$ at absolute zero; $f(\mu) = \frac{1}{2}$.
- Confusing activity with the number of nuclei.
- Judging stability by total binding energy instead of binding energy per nucleon.
- Writing beta decay without the antineutrino.

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Key terms

band gap: The range of energies with no electron states between the valence and conduction bands.

doping: Adding small amounts of impurity to change the number of charge carriers.

Fermi level: The energy at which a state has a 50% chance of being occupied.

activity: The number of decays per second in a sample.

half-life: The time for half of the nuclei in a sample to decay.

binding energy: The energy needed to separate a nucleus into its protons and neutrons.

Check yourself

- What fraction of a sample remains after five half-lives?
- What is the Fermi–Dirac occupancy of a state at $E = \mu + kT$?
- A π^- decays to a μ^- and one more particle. Which?

Answers: $1/32$. $1/(e+1) = 0.27$. A muon antineutrino $\bar{\nu}_\mu$, which keeps the muon-lepton number at zero.

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Hamiltonian mechanics and astrophysical scaling

GRE Subject Test · GRE Physics

In 1838 Friedrich Bessel measured the tiny yearly shift of the star 61 Cygni against more distant stars: about 0.3 arcseconds. That parallax put the star about 10 light-years away, the first distance to a star ever measured. Once the distance is known, the brightness we receive tells us the power the star gives out, its **luminosity** 光度. Much of astrophysics is scaling laws like this one, and mechanics supplies many of them.

GRE Physics questions on advanced mechanics and astrophysics ask you to:

- Build the **Hamiltonian** 哈密顿量 from a Lagrangian and use Hamilton's equations.
- Know when the Hamiltonian is conserved and when it equals the energy.
- Scale orbital speed, period and angular momentum with radius.
- Use $L = 4\pi R^2 \sigma T^4$, the inverse-square law and Wien's law.

From Lagrangian to Hamiltonian

The **canonical momentum** 正则动量 of a coordinate q is $p = \partial L / \partial \dot{q}$. The Hamiltonian is a Legendre transform, written in q and p instead of q and $\dot{q}$:

$$H(q, p) = p\dot{q} - L$$

- Oscillator: $L = \tfrac{1}{2} m\dot{q}^2 - \tfrac{1}{2} k q^2$ gives $p = m\dot{q}$ and $H = \dfrac{p^2}{2m} + \tfrac{1}{2} k q^2$.
- In a magnetic field, the canonical momentum $m\mathbf{v} + q\mathbf{A}$ is not the same as $m\mathbf{v}$.

Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

- Oscillator: $\dot{q} = p/m$ and $\dot{p} = -kq$, so $\ddot{q} = -(k/m)q$, as Newton gives.
- If H does not contain q, then p is conserved.
- If H has no explicit time dependence, H is conserved. It equals $T + V$ when the constraints do not depend on time and V does not depend on velocity; otherwise it may not.

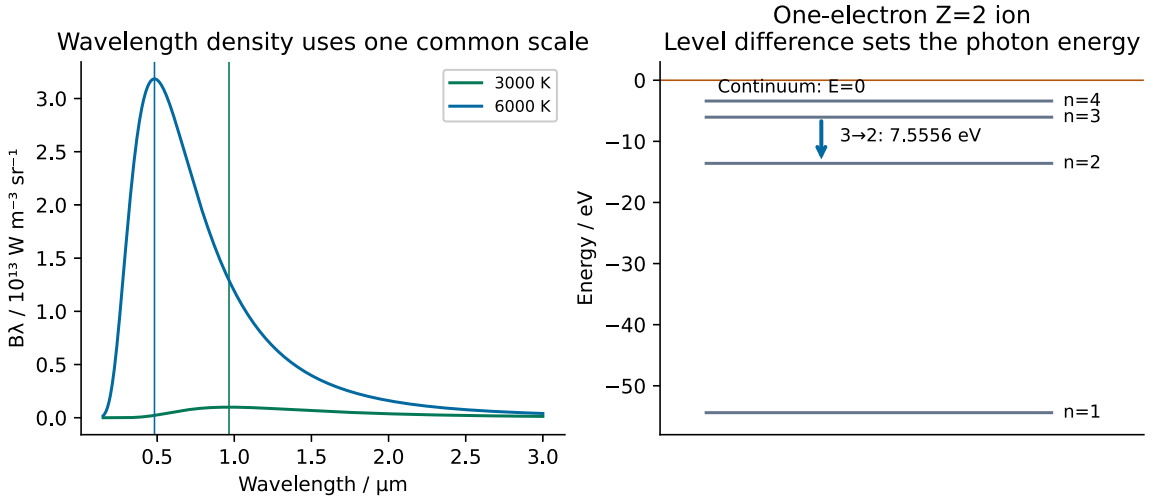
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Circular orbits

Gravity provides the centripetal force: $GMm/r^2 = mv^2/r$. Each orbital quantity then has its own power of r.

$$v = \sqrt{\frac{GM}{r}}, \qquad T^2 = \frac{4\pi^2 r^3}{GM}, \qquad L = m\sqrt{GMr}$$

- $T^2 \propto r^3$ is **Kepler's third law** 开普勒第三定律.
- Total energy is $-GMm/2r$: a lower orbit is faster but has less energy.



Doubling the temperature halves the peak wavelength; for a $Z = 2$ ion every level is $Z^2 = 4$ times the hydrogen value

Stars: luminosity and flux

A star behaves roughly like a **blackbody** 黑体. Its luminosity depends on its size and surface temperature, and the **flux** 通量 we receive falls with the square of the distance.

$$L = 4\pi R^2 \sigma T^4, \qquad F = \frac{L}{4\pi d^2}$$

- Twice the radius at the same temperature: four times the luminosity. Twice the temperature: sixteen times.
- Twice as far away: a quarter of the flux. That is why **parallax** 视差 distances matter.

Wien's law and spectra

The peak of a blackbody spectrum moves to shorter wavelength as the temperature rises: $\lambda_{\max} T \approx 2.9 \times 10^{-3}$ m K.

- The Sun (5800 K) peaks near 500 nm; a 3000 K red dwarf peaks near 970 nm, in the infrared.
- Spectral lines add composition: a one-electron ion with $Z = 2$ has every level $Z^2 = 4$ times deeper than hydrogen, so its $3 \to 2$ photon is $4 \times 1.89 = 7.56$ eV.

Worked example: a higher satellite

An identical satellite orbits at 9 times the radius of another.

- Speed: $v \propto r^{-1/2}$, so it moves at 1/3 of the speed.
- Angular momentum: $L \propto r^{1/2}$, so it has 3 times as much.
- Period: $T \propto r^{3/2}$, so its year is $9^{3/2} = 27$ times longer.

Worked example: sunlight at the Earth

The Sun's luminosity is 3.8×10^{26} W and the Earth is 1.5×10^{11} m away.

- Flux: $F = \dfrac{3.8 \times 10^{26}}{4\pi (1.5 \times 10^{11})^2} = 1.3 \times 10^{3}$ W/m^2, close to the measured 1361 W/m^2.
- Peak wavelength: $\lambda_{\max} = 2.9 \times 10^{-3} / 5800 = 5.0 \times 10^{-7}$ m, green light.
- A star of the same luminosity but 3 times farther away would give 1/9 of this flux.

Common errors

- Writing $H = T + V$ for every system, including time-dependent constraints.
- Using the same power of r for speed, period and angular momentum.
- Confusing luminosity (power given out) with flux (power received per square metre).
- Forgetting the fourth power of temperature in L.
- Using degrees Celsius in Wien's law.

Key terms

luminosity: The total power a star gives out.

Hamiltonian: The function of coordinates and momenta that generates the equations of motion.

canonical momentum: $p = \partial L / \partial \dot{q}$, the momentum paired with a coordinate.

Kepler's third law: For orbits round one body, T^2 is proportional to r^3.

blackbody: An ideal body that absorbs all radiation and gives out a spectrum set

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only by its temperature.

flux: Power received per unit area.

parallax: The apparent shift of a nearby star as the Earth moves round the Sun.

Check yourself

- A star has twice the Sun's radius and the same surface temperature. How does its luminosity compare?
- Two stars have the same luminosity; one is 3 times farther away. How do their fluxes compare?
- For a free particle, $H = p^2/2m$. Which quantity is conserved, and why?

Answers: 4 times. The farther one gives 1/9 of the flux. Momentum, because H does not contain q.

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Collisions, work and oscillator energy

GRE Subject Test · GRE Physics

In 1668 the Royal Society in London asked for the laws of colliding bodies. Three answers arrived: from John Wallis, Christopher Wren and Christiaan Huygens. All three agreed that the total "quantity of motion", counted with direction, stays the same. Huygens found something more: in a perfectly elastic collision, the sum of mv^2 for all the bodies is unchanged too. These are momentum and kinetic energy, and the first is conserved in every collision, the second only in some.

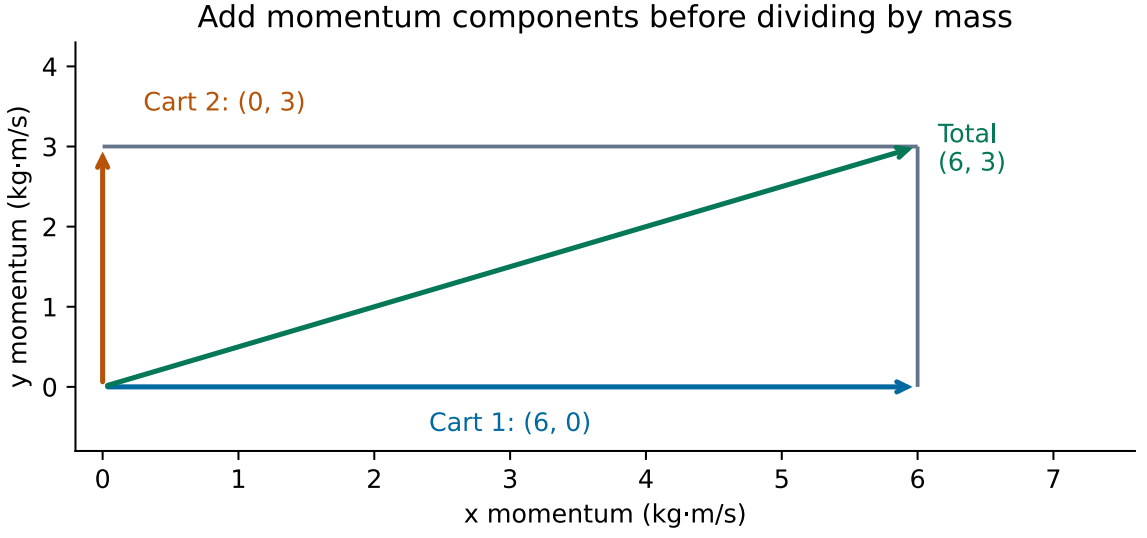
GRE Physics mechanics questions on collisions and energy ask you to:

- Conserve momentum as a vector, component by component.
- Tell an **elastic collision** 弹性碰撞 from an **inelastic collision** 非弹性碰撞.
- Use the **work–energy theorem** 动能定理 and spring energy.
- Find oscillator energy, maximum speed and pendulum periods.

Momentum by components

During a short collision, external forces give almost no **impulse** 冲量, so the total momentum is conserved. It is a vector: conserve each component separately and take the magnitude only at the end.

- A 2 kg cart at 3 m/s east and a 1 kg cart at 3 m/s north have total momentum $(6, 3)$ kg m/s.
- If they stick, the 3 kg wreck moves at $(2, 1)$ m/s, about 2.2 m/s.
- Adding the magnitudes, $6 + 3 = 9$ kg m/s, would be wrong.



Add momentum as vectors first, then divide by the total mass

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Elastic and inelastic collisions

In an elastic collision, kinetic energy is conserved as well as momentum. In an inelastic one, some becomes **internal energy** 内能; it is lost most when the bodies stick together.

- One-dimensional elastic collision with m_2 at rest: $v_1' = \dfrac{m_1 - m_2}{m_1 + m_2} v_1$ and $v_2' = \dfrac{2m_1}{m_1 + m_2} v_1$.
- Equal masses swap velocities; a light ball bounces back off a heavy one.
- In an elastic collision, the relative speed of approach equals the relative speed of separation.

Work–energy theorem and springs

The net work done on a body equals its change in kinetic energy: $\Delta K = \int \mathbf{F}_{\text{net}} \cdot d\mathbf{r}$.

- Doubling the speed quadruples K, so with the same braking force the stopping distance is four times as long.
- A spring stores $U = \tfrac{1}{2} k x^2$.
- Same force F on two springs: $x = F/k$, so $U = F^2/2k$ and the softer spring stores more. Same extension: the stiffer spring stores more.

Oscillator energy

In simple harmonic motion, energy moves between kinetic and potential, but the total is fixed:

$$E = \tfrac{1}{2} m v^2 + \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2 = \tfrac{1}{2} m v_{\max}^2$$

- So $v_{\max} = \omega A$, with $\omega = \sqrt{k/m} = 2\pi f$.
- At $x = A/2$, the potential energy is $E/4$ and the kinetic energy $3E/4$.

Pendulums and small angles

A simple pendulum obeys $\ddot{\theta} + (g/L)\sin\theta = 0$. For small angles, $\sin\theta \approx \theta$:

$$T = 2\pi \sqrt{\frac{L}{g}}$$

- The period does not depend on the mass or (for small swings) the **amplitude** 振幅.
- Four times the length doubles the period. Large amplitudes make the period slightly longer.

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Worked example: a two-dimensional sticking collision

For the two carts above:

- Momentum: $(6,0)+(0,3)=(6,3)$ kg m/s, so $\mathbf{v}=(6,3)/3=(2,1)$ m/s.
- Kinetic energy before: $\tfrac{1}{2}(2)(9)+\tfrac{1}{2}(1)(9)=13.5$ J. After: $\tfrac{1}{2}(3)(2^2+1^2)=7.5$ J.
- So 6.0 J becomes internal energy: heat, sound and bent metal.

Worked example: a one-dimensional elastic collision

A 1.0 kg ball at 4.0 m/s hits a 3.0 kg ball at rest, elastically.

- $v_1'=\dfrac{1-3}{4}(4.0)=-2.0$ m/s (it bounces back); $v_2'=\dfrac{2}{4}(4.0)=2.0$ m/s.
- Momentum check: $4.0=-2.0+3.0\times 2.0$.
- Energy check: $\tfrac{1}{2}(1)(16)=8.0$ J before; $\tfrac{1}{2}(1)(4)+\tfrac{1}{2}(3)(4)=8.0$ J after.

Common errors

- Adding momentum magnitudes instead of vectors.
- Assuming kinetic energy is conserved because momentum is.
- Saying the stiffer spring always stores more energy; it depends on whether force or extension is fixed.
- Including the mass in the pendulum period.
- Using $T=2\pi\sqrt{L/g}$ for large swings.

Key terms

elastic collision: A collision in which kinetic energy is conserved.

inelastic collision: A collision in which some kinetic energy becomes internal energy.

work–energy theorem: The net work done on a body equals its change in kinetic energy.

impulse: Force multiplied by the time it acts; the change in momentum.

internal energy: Energy stored in the random motion and arrangement of particles.

amplitude: The largest displacement from equilibrium.

Check yourself

- A 0.50 kg oscillator passes equilibrium at 0.20 m/s. What is its total energy?
- A pendulum is made four times as long. What happens to its period?
- A 10 N force stretches springs of $k=100$ N/m and $k=400$ N/m. How much energy does each store?

Answers: $\tfrac{1}{2}(0.50)(0.20)^2=0.010$ J. It doubles. $F^2/2k$: 0.50 J and 0.125 J.

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Rotation, buoyancy and terminal-speed balances

GRE Subject Test · GRE Physics

Rotation, buoyancy and terminal speed

On 2 August 1971, at the end of the Apollo 15 moonwalk, astronaut David Scott held a 1.32 kg geological hammer in one hand and a 0.03 kg falcon feather in the other, and let go. With no air on the Moon, they landed together, as Galileo had predicted. On Earth the feather would drift down slowly: air **drag** 阻力 balances its weight at a very low terminal speed. Many GRE mechanics problems are about these "other" forces and how they share motion: torque, rolling, buoyancy and drag.

GRE Physics questions on rotation and fluids ask you to:

- Use **torque** 力矩 and **moment of inertia** 转动惯量 about the right axis.
- Share kinetic energy between translation and rotation in rolling.
- Apply Archimedes' principle to floating and composite objects.
- Find **terminal speed** 终端速度 from the drag law.

Torque about a pivot

The rotational form of Newton's law is $\tau = I\alpha$, with torque $\tau = rF\sin\phi$ and I taken about the same pivot.

- Uniform rod about its centre: $\frac{1}{12}ML^2$. About one end, by the **parallel axis** 平行轴 rule $I = I_{\text{cm}} + Md^2$: $\frac{1}{12}ML^2 + M(L/2)^2 = \frac{1}{3}ML^2$.
- A rod pivoted at one end, at angle θ from the vertical: $\tau = Mg\frac{L}{2}\sin\theta$, so $\alpha = \dfrac{3g\sin\theta}{2L}$.
- Other shapes about their centres: hoop MR^2, disk $\frac{1}{2}MR^2$, solid sphere $\frac{2}{5}MR^2$.

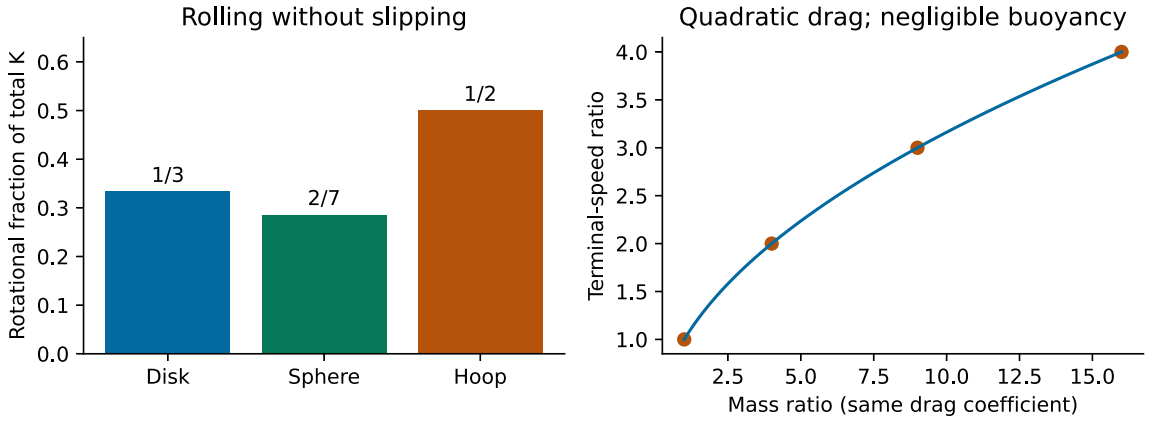
Rolling without slipping

When a body rolls without slipping, $v = R\omega$. Write $I = \beta MR^2$; then the kinetic energy is $K = \frac{1}{2}(1+\beta)Mv^2$.

$$v^2 = \frac{2gh}{1+\beta}$$

- The final speed after a drop h does not depend on M or R, only on β.
- The share of energy in rotation is $\beta/(1+\beta)$: 1/3 for a disk, 2/7 for a sphere, 1/2 for a hoop.
- In a race down a slope, the smallest β wins: sphere, then disk, then hoop.

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Rolling shares energy by $I/(mr^2)$; quadratic drag gives terminal speed $\propto \sqrt{m}$

Buoyancy

Buoyancy 浮力 is the upward force of a fluid, equal to the weight of the fluid displaced (Archimedes' principle).

- A floating object displaces its own weight of fluid. Wood of density 600 kg/m^3 floats in water with 60% of its volume below the surface.
- Count every submerged part. For a composite object, the total displaced volume is set by the total mass, wherever the parts are.

Terminal speed

At terminal speed the acceleration is zero: drag balances the weight (minus buoyancy, if it matters).

- Quadratic drag (fast or large objects), $Cv^2 = Mg$: $v_t = \sqrt{Mg/C}$. Doubling the mass multiplies v_t by $\sqrt{2}$.
- Linear drag (slow, small objects in viscous fluids), $bv = Mg$: $v_t \propto M$.
- With buoyancy, replace Mg by $(M - \rho_f V)g$.

Worked example: a falling rod

A uniform rod 1.2 m long is pivoted at one end and released from horizontal.

- Torque: $\tau = Mg(L/2)$; moment of inertia: $\frac{1}{3}ML^2$.
- $\alpha = \dfrac{3g}{2L} = \dfrac{3(9.8)}{2(1.2)} = 12.3$ rad/s^2.
- The free end starts with acceleration $\alpha L = 14.7$ m/s^2, more than g. A coin resting on that end would be left behind.

Worked example: flipping a composite block

A wooden block carries a small stone of volume $0.10V$ glued on top, and floats with $0.60V$ of wood under water. Now turn it over so the stone hangs underneath.

- Weight is unchanged, so the total displaced volume is still $0.60V$.
- The submerged stone now displaces $0.10V$ itself.
- So only $0.50V$ of wood is under water: the block floats higher.

Common errors

- Taking I about the centre when the body turns about a pivot.
- Thinking a heavier or larger ball rolls down faster; only the shape factor β matters.
- Ignoring the submerged parts of a composite object.
- Using $v_t \propto M$ for quadratic drag.
- Forgetting buoyancy for objects falling through dense liquids.

Key terms

drag: The resistive force a fluid exerts on a moving object.

torque: The turning effect of a force, $rF\sin\phi$.

moment of inertia: Resistance to angular acceleration, $\sum mr^2$ about an axis.

parallel axis: The rule $I = I_{\text{cm}} + Md^2$ for an axis parallel to one through the centre of mass.

terminal speed: The steady speed at which drag balances the driving force.

buoyancy: The upward force from a fluid, equal to the weight of fluid displaced.

Check yourself

- A hoop and a solid sphere race from rest down the same slope. Which wins, and what is the sphere's v^2 after a drop h?
- With quadratic drag, how does the terminal speed change if the mass is doubled?
- A plastic of density 900 kg/m^3 floats in water. What fraction is submerged?

Answers: the sphere, with $v^2 = 2gh/(1 + 2/5) = 10gh/7$. It rises by a factor of $\sqrt{2}$. 90%.

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Coupled acceleration and relative motion

GRE Subject Test · GRE Physics

In 1784 George Atwood, a Cambridge lecturer, described a machine for showing Newton's laws. Two nearly equal masses hung from a light pulley on one string. Because their weights almost cancelled, they accelerated slowly enough to time with a pendulum clock, and students could check that the acceleration was constant. Atwood's machine is still the model for every **pulley** 滑轮 problem: one string, one acceleration, a **tension** 张力 found from the equations.

GRE Physics questions on coupled bodies and frames ask you to:

- Use the **constraint** 约束 of a string to link accelerations.
- Write Newton's law for each body separately.
- Handle pulleys with mass.
- Change between **reference frames** 参考系 with **relative velocity** 相对速度.

One string, one acceleration

A light, inextensible string makes every body on it move with the same speed and the same size of acceleration. Draw each body on its own: tension is internal to the system, but external to each body.

- Mass m_1 on a smooth table pulled by a hanging mass m_2: $T = m_1 a$ and $m_2 g - T = m_2 a$.
- So $a = \dfrac{m_2 g}{m_1 + m_2}$ and $T = \dfrac{m_1 m_2 g}{m_1 + m_2}$.
- T is less than $m_2 g$; if they were equal, m_2 could not accelerate.

The Atwood machine

Two masses $m_1 < m_2$ hang over an ideal pulley.

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2}, \qquad T = \frac{2 m_1 m_2 g}{m_1 + m_2}$$

- Nearly equal masses give a small acceleration, which is why Atwood used them.
- The tension lies between the two weights.

When the pulley is not ideal

Equal tension on both sides needs a massless, frictionless pulley. A pulley of radius R and moment of inertia I needs its own equation.

- $(T_2 - T_1)R = I\alpha$ and $a = R\alpha$.
- The pulley adds I/R^2 to the "inertia" in the denominator: $a = \dfrac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}$.

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Relative velocity

The velocity of an object relative to a moving frame is $\mathbf{v}_{\text{rel}} = \mathbf{v}_{\text{ground}} - \mathbf{V}_{\text{frame}}$, component by component.

- A payload released from an aircraft flying level at 80 m/s keeps that horizontal velocity. After 3 s its ground velocity is $(80, -30)$ m/s, but relative to the aircraft it is $(0, -30)$ m/s: seen from the aircraft, it drops straight down.
- A boat heading straight across a river at 4 m/s relative to water that flows at 3 m/s moves at 5 m/s over the ground.

Subtract aircraft velocity; components in m/s

Velocity after 3 s	Horizontal	Vertical
Ground	80	−30
Aircraft	80	0
Payload relative to aircraft	0	−30

Subtract the aircraft's velocity component by component

Check the limits

A good answer behaves sensibly in extreme cases.

- Table pulley: $m_1 \to 0$ gives $a \to g$, $T \to 0$; $m_1 \to \infty$ gives $a \to 0$, $T \to m_2 g$.
- Energy check without friction: $m_2 g s = \tfrac{1}{2}(m_1 + m_2)v^2$ agrees with $v^2 = 2as$.

Worked example: a table and a pulley

$m_1 = 3.0$ kg on a smooth table; $m_2 = 2.0$ kg hangs; take $g = 10$ m/s^2.

- $a = \dfrac{2.0 \times 10}{5.0} = 4.0$ m/s^2 and $T = 3.0 \times 4.0 = 12$ N, less than the 20 N weight.
- After falling 0.50 m from rest: $v^2 = 2(4.0)(0.50)$, so $v = 2.0$ m/s.
- Energy check: $2.0 \times 10 \times 0.50 = 10$ J $= \tfrac{1}{2}(5.0)(2.0)^2$.

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Worked example: a pulley with mass

Masses of 2.0 kg and 3.0 kg hang over a pulley that is a uniform disk of mass 2.0 kg ($I = \frac{1}{2}MR^2$); $g = 10$ m/s^2.

- The pulley adds $I/R^2 = \frac{1}{2}(2.0) = 1.0$ kg.
- $a = \dfrac{(3.0 - 2.0)(10)}{2.0 + 3.0 + 1.0} = 1.7$ m/s^2, compared with 2.0 m/s^2 for an ideal pulley.
- The tensions now differ: $T_1 = 2.0(10 + 1.7) = 23$ N and $T_2 = 3.0(10 - 1.7) = 25$ N.

Common errors

- Setting the tension equal to the hanging weight.
- Using one equation for the whole system and then trying to find the tension from it.
- Assuming equal tensions on both sides of a pulley that has mass.
- Adding frame velocities instead of subtracting them.
- Using the energy check when friction acts.

Key terms

pulley: A wheel that changes the direction of a string's pull.

tension: The pulling force a string or rope exerts.

constraint: A condition that links the motions of bodies, such as a fixed string length.

reference frame: The viewpoint and coordinates from which motion is measured.

relative velocity: The velocity of an object as measured in a moving frame.

Check yourself

- An Atwood machine has masses of 4.0 kg and 6.0 kg; $g = 10$ m/s^2. Find a and T.
- Rain falls vertically at 8.0 m/s. A car drives at 6.0 m/s. How fast does the rain hit the windscreen, as seen from the car?
- In the table pulley, what happens to T when m_1 becomes very large?

Answers: $a = 2.0$ m/s^2 and $T = 2(4)(6)(10)/10 = 48$ N. 10 m/s, at 37° from the vertical ($\tan\theta = 6/8$). $T \to m_2 g$, because the system barely accelerates.

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Electrostatic superposition, flux and conductors

GRE Subject Test · GRE Physics

In 1785 Charles-Augustin de Coulomb hung a light rod from a thin silver wire, with a small charged ball on one end. When he brought another charged ball near, the rod turned and twisted the wire. From the angle of twist he found that the force between two charges falls as the inverse square of their distance. Every electrostatics problem builds on his law, added up over many charges by **superposition** 叠加.

GRE Physics electrostatics questions ask you to:

- Add electric fields as vectors and potentials as numbers.
- Find the **flux** 通量 through open as well as closed surfaces.
- Use **electrostatic equilibrium** 静电平衡 in conductors and **induced charge** 感应电荷.
- Recognise the method of images.

Coulomb's law and superposition

The field of a set of point charges is the vector sum of each charge's field:

$$\mathbf{E} = \sum_i \frac{Q_i(\mathbf{r} - \mathbf{r}_i)}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}_i|^3}$$

- Resolve each field into components, add, then find the size. Forces of 3 N to the right and 4 N upwards combine to 5 N, not 7 N.
- Potential adds as a scalar: $V = \sum Q_i/(4\pi\varepsilon_0 r_i)$. This is easier, but $V = 0$ does not mean $\mathbf{E} = 0$.

Flux through open surfaces

Gauss's law applies to closed surfaces. For an open surface, use symmetry to share out the total flux q/ε_0.

- A point charge at the centre of a cube sends $q/(6\varepsilon_0)$ through each face.
- A point charge beside an infinite plane sends half its flux through the plane, $q/(2\varepsilon_0)$, whatever the distance.

Conductors in equilibrium

In equilibrium, charges in a conductor stop moving, so the field inside the metal is zero and the whole conductor is **equipotential** 等势的.

- Any net charge lies on the surfaces.
- The potential of the metal is constant, but not usually zero.

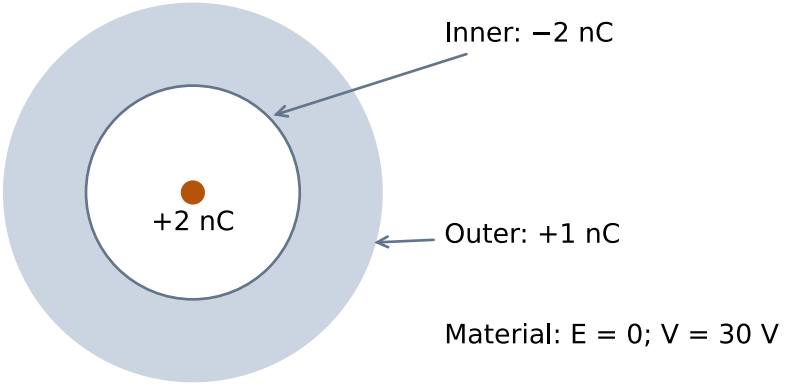
- Just outside, the field is perpendicular to the surface.

A charge inside a shell

Put a charge Q in the cavity of a conducting shell (inner radius a, outer radius b, net charge q). Gauss's law with a surface inside the metal, where $E = 0$, shows the enclosed charge must be zero.

- The inner surface carries $-Q$; the outer surface carries $q + Q$.
- Cavity ($r < a$): $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$. Metal: $E = 0$. Outside ($r > b$): $E = \dfrac{Q+q}{4\pi\varepsilon_0 r^2}$.
- V is continuous everywhere; E jumps at each charged surface. An off-centre charge still induces $-Q$ inside, but the cavity field is then not radial.

Shell net charge −1 nC; outer radius 0.30 m



Induced charges make $E = 0$ in the metal; the shell's potential is $k(+1\ \text{nC})/0.30\ \text{m} = 30\ \text{V}$

The method of images

The **method of images** 镜像法 replaces a grounded conducting plane by an imaginary opposite charge, placed like a mirror image. The real charge and its image together give $V = 0$ on the plane.

- A charge q at height d above a grounded plane feels the force of $-q$ at distance $2d$: $F = \dfrac{q^2}{4\pi\varepsilon_0(2d)^2}$, towards the plane.
- The image is used only to find the field above the plane; there is no field inside the conductor.

Worked example: the shell at 30 V

A $+2.0$ nC charge sits at the centre of a shell of net charge -1.0 nC and outer radius $b = 0.30$ m.

- Inner surface: -2.0 nC. Outer surface: $-1.0 - (-2.0) = +1.0$ nC.
- Potential of the metal: $V = \dfrac{(9.0 \times 10^{9})(1.0 \times 10^{-9})}{0.30} = 30$ V.
- Field in the metal: zero, even though the potential is 30 V.

Worked example: two equal charges

Charges of $+1.0$ nC sit at $(0, 0)$ and $(0.60, 0)$ m. Find $\mathbf{E}$ and V at $(0.30, 0.40)$ m.

- Each charge is 0.50 m away, giving $E = (9.0 \times 10^{9})(1.0 \times 10^{-9})/0.25 = 36$ N/C each.
- The x-components cancel; the y-components add: $2 \times 36 \times 0.80 = 58$ N/C upwards.
- $V = 2 \times (9.0)/0.50 = 36$ V.

Common errors

- Adding field magnitudes without resolving into components.
- Thinking that $E = 0$ inside a conductor means $V = 0$.
- Using Gauss's law on an open surface without a symmetry argument.
- Putting the induced $-Q$ on the outer surface.
- Forgetting that the image charge sits at $2d$ from the real one, not d.

Key terms

superposition: The total field of several charges is the vector sum of their separate fields.

flux: The amount of field passing through a surface.

electrostatic equilibrium: The state in which no charges move in a conductor, so $E = 0$ inside it.

induced charge: Charge that gathers on a conductor's surface because of nearby charges.

equipotential: At the same potential everywhere.

method of images: Replacing a conductor by imaginary charges that give the same boundary conditions.

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Check yourself

- Forces of 3.0 N and 4.0 N act at right angles on a charge. What is the net force?
- A point charge sits at the centre of a cube. What flux passes through one face?
- What is the field inside the metal of the 30 V shell, and what is the potential there?

Answers: 5.0 N. $q/(6\varepsilon_0)$. $E = 0$ and $V = 30$ V.

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Circuit power, induction and charged-particle motion

GRE Subject Test · GRE Physics

In 1931 Ernest Lawrence and his student M. Stanley Livingston, at Berkeley, built a particle accelerator small enough to hold in one hand: a flat brass box about 11 cm across, placed between the poles of a magnet. Protons circled inside and were given a small electric kick twice on each turn. The trick worked because a charge in a magnetic field goes round at a frequency that does not depend on its speed. Faster protons simply moved on wider circles. Their **cyclotron** 回旋加速器 reached 80 keV.

GRE Physics questions on circuits and magnetism ask you to:

- Reduce resistor networks and find each branch's voltage and current.
- Use rms values, **impedance** 阻抗 and resonance in AC circuits.
- Apply Faraday's law to coils.
- Follow charges in magnetic fields: radii, cyclotron frequency, the **velocity selector** 速度选择器.

Reduce, then expand

Series resistances add; for parallel resistors, the conductances $1/R$ add. Reduce the network to one resistor, find the total current, then work back out.

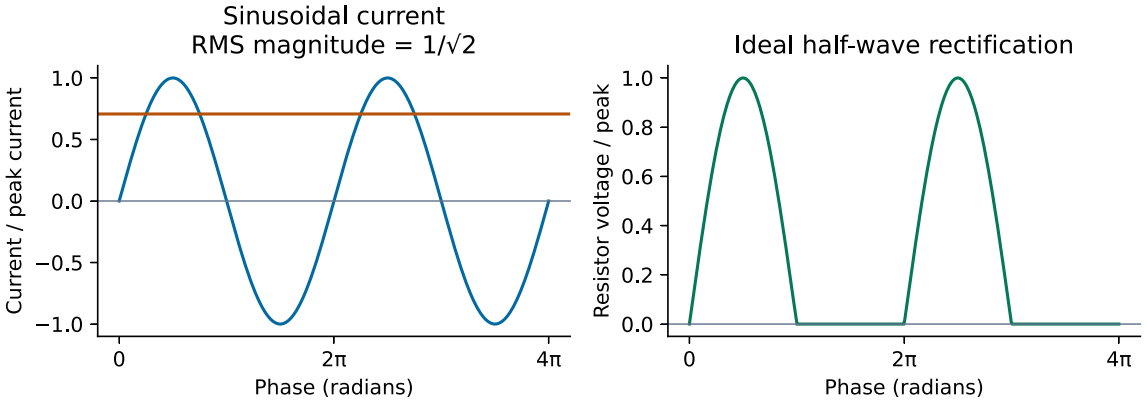
- Kirchhoff's current rule: the currents into a node sum to zero.
- Kirchhoff's voltage rule: the voltage changes round any loop sum to zero.
- Not every resistor "sees" the battery voltage; series resistors share it.

AC power

For a sinusoidal current, the **rms** 均方根 value is the peak divided by $\sqrt{2}$. Average power uses rms values:

$$P = I_{\text{rms}}^2 R = V_{\text{rms}} I_{\text{rms}} \cos\varphi$$

- Ideal inductors and capacitors take no average power.
- Series RLC: $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$. At **resonance** 共振, $\omega_0 = 1/\sqrt{LC}$, $|Z| = R$ and the current is largest.
- A single ideal **diode** 二极管 passes only half of each cycle: half-wave rectification.



RMS of a sine is peak/√2; a half-wave rectifier passes only the positive half-cycles

Faraday's law in a coil

Flux through one turn is $\Phi = BA\cos\theta$, with θ measured from the normal to the coil. For N turns:

$$\mathcal{E} = -N\frac{d\Phi}{dt}$$

- The induced current opposes the change in flux, not the field itself.
- A coil spinning at ω in a steady field gives $\mathcal{E} = NBA\omega\sin\omega t$: a generator.

Magnetic fields of currents

- Long straight wire: $B = \dfrac{\mu_0 I}{2\pi r}$.
- Centre of a circular loop: $B = \dfrac{\mu_0 I}{2R}$, times N for a flat coil.
- Long solenoid: $B = \mu_0 n I$ inside, with n turns per metre, and almost zero outside.

Charges in magnetic fields

The magnetic force $q\mathbf{v}\times\mathbf{B}$ is perpendicular to the velocity, so it changes direction but not speed. A charge moving across a uniform field goes in a circle:

$$r = \frac{mv}{|q|B}, \qquad \omega_c = \frac{|q|B}{m}$$

- ω_c does not depend on speed. Deuterons and alpha particles both have half the proton's ω_c, since q/m is halved.
- Velocity selector: crossed $\mathbf{E}$ and $\mathbf{B}$ let through undeflected only charges with $qE = qvB$, so $v = E/B$.

Worked example: a resistor network

A 24 V battery drives a 2 Ω resistor in series with 3 Ω and 6 Ω in parallel.

- $3 \parallel 6 = 2\ \Omega$, so the total is 4 Ω and the current is 6 A.
- The 2 Ω resistor takes 12 V; the parallel pair has 12 V, so the branch currents are 4 A and 2 A.
- Power: 72 W, 48 W and 24 W, totalling 144 W $= 24 \times 6$.

Worked example: a cyclotron

A proton circles in a 1.5 T field.

- Frequency: $f = \dfrac{qB}{2\pi m} = \dfrac{(1.6 \times 10^{-19})(1.5)}{2\pi(1.67 \times 10^{-27})} = 23$ MHz.
- At a radius of 0.50 m: $v = qBr/m = 7.2 \times 10^{7}$ m/s.
- Kinetic energy: $\tfrac{1}{2}mv^2 = 4.3 \times 10^{-12}$ J $= 27$ MeV. At $v = 0.24c$, relativity changes this by only a few per cent.

Common errors

- Giving every resistor the full battery voltage.
- Halving the power again after using rms values.
- Measuring the flux angle from the plane of the coil instead of its normal.
- Thinking a magnetic field speeds up a charge; it only turns it.
- Thinking the cyclotron frequency grows with speed (it does not, until relativity matters).

Key terms

cyclotron: An accelerator in which charges circle in a magnetic field and are pushed by an alternating voltage.

impedance: The AC opposition to current, combining resistance and reactance.

velocity selector: Crossed electric and magnetic fields that pass charges of one speed only.

rms: Root mean square; for a sine wave, the peak value divided by $\sqrt{2}$.

resonance: The frequency at which an RLC circuit's reactances cancel and the current is largest.

diode: A component that lets current flow in one direction only.

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Check yourself

- The mains voltage has a peak of 170 V. What is its rms value?
- A velocity selector has $E = 6000$ V/m and $B = 0.20$ T. Which speed passes?
- How does an alpha particle's cyclotron frequency compare with a proton's in the same field?

Answers: $170/\sqrt{2} = 120$ V. $v = E/B = 3.0 \times 10^4$ m/s. Half: twice the charge, four times the mass.

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Maxwell waves, energy flow and boundary conditions

GRE Subject Test · GRE Physics

In 1862 James Clerk Maxwell worked out the speed of waves in his model of electric and magnetic fields. It depended only on electrical constants that Wilhelm Weber and Rudolf Kohlrausch had measured in 1856 with capacitors and currents. The result was about 3.1×10^8 m/s, very close to the speed of light that Hippolyte Fizeau had measured in 1849. Maxwell concluded that light is an electromagnetic wave. The key step was a new term in Ampère's law: the **displacement current** 位移电流.

GRE Physics questions on Maxwell's equations ask you to:

- Use the displacement current in Ampère's law.
- Find the speed, impedance and field ratio of waves in a medium.
- Get directions from $\mathbf{E} \times \mathbf{B}$, the **Poynting vector** 坡印廷矢量.
- Apply **boundary conditions** 边界条件 at surfaces.

Displacement current

While a capacitor charges, no charge crosses the gap between its plates, yet a magnetic field circles the gap as if the current continued. Maxwell's extra term makes Ampère's law work for any surface on the same loop:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(I + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$

- Between the plates, $I_d = \varepsilon_0 A \, dE/dt$ equals the current in the wires.
- It is a rate of change of electric flux, not a flow of charge.

Waves in a medium

In a medium with **permittivity** 介电常数 ε and permeability μ:

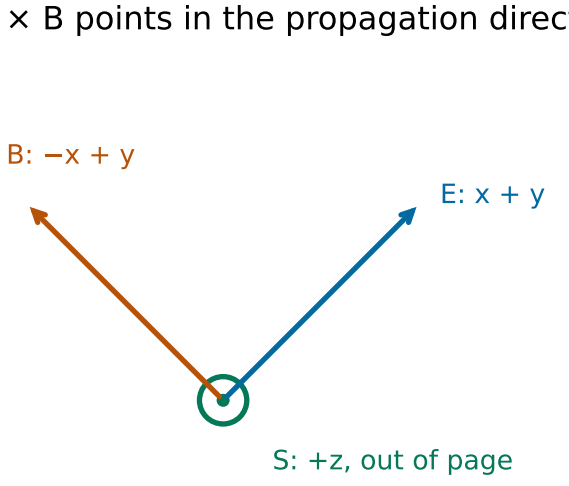
$$v = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad n = \frac{c}{v} = \sqrt{\mu_r \varepsilon_r}, \qquad \frac{E}{B} = v$$

- For a non-magnetic medium, $v = c/\sqrt{\varepsilon_r}$: $\varepsilon_r = 9$ gives $v = 1.0 \times 10^8$ m/s and $n = 3$.
- Use the permittivity at the wave's frequency. Water has $\varepsilon_r \approx 80$ for static fields, but $n = 1.33$ for light.

Directions of E, B and S

The fields are perpendicular to each other and to the direction of travel $\hat{\mathbf{n}}$: $\mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}/v$.

- The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ points along the energy flow, which is the direction of travel.
- A wave in $kz - \omega t$ travels towards $+z$; one in $kz + \omega t$ travels towards $-z$.
- Travelling along $+z$ with $\mathbf{E} \parallel \hat{x} + \hat{y}$: $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{z} \times \hat{y} = -\hat{x}$, so $\mathbf{B} \parallel -\hat{x} + \hat{y}$.



$E \times B$ gives the direction in which the wave travels

Energy and intensity

The **intensity** 强度 of a wave is its average power per unit area:

$$I = \langle S \rangle = \tfrac{1}{2} c \varepsilon_0 E_0^2$$

- Light carries momentum too, so it exerts **radiation pressure** 辐射压: I/c on an absorbing surface, $2I/c$ on a perfect mirror.
- Intensity from a point source falls as $1/r^2$, so E_0 falls as $1/r$.

Boundary conditions

At a surface between two media, Maxwell's equations fix how the fields change:

- Normal $\mathbf{B}$ is continuous ($\nabla \cdot \mathbf{B} = 0$); tangential $\mathbf{E}$ is continuous.
- Normal $\mathbf{D}$ jumps by the free surface charge; tangential $\mathbf{H}$ jumps by the free surface current.
- At an ideal superconductor ($B = 0$ inside), the field just outside has no normal component: it runs along the surface. At a perfect conductor, tangential E is zero.

Worked example: a charging capacitor

A parallel-plate capacitor with plates of area 0.010 m^2 is charged by a steady 2.0 A current.

- $dE/dt = \dfrac{I}{\varepsilon_0 A} = \dfrac{2.0}{(8.85\times10^{-12})(0.010)} = 2.3\times10^{13}$ V/(m s).
- Displacement current: $\varepsilon_0 A\, dE/dt = 2.0$ A, equal to the wire current.
- So a loop round the gap, 10 cm from the axis (outside circular plates of radius 5.6 cm), has $B = \mu_0 I/(2\pi r) = 4.0\ \mu$T, the same as round the wire.

Worked example: a laser beam

A 1.0 mW laser beam has a cross-section of 1.0 mm^2.

- Intensity: $I = 10^{-3}/10^{-6} = 1.0\times10^{3}$ W/m^2.
- $E_0 = \sqrt{2I/(c\varepsilon_0)} = \sqrt{2000/(2.66\times10^{-3})} = 870$ V/m and $B_0 = E_0/c = 2.9\ \mu$T.
- Pressure on an absorber: $I/c = 3.3\times10^{-6}$ Pa.

Common errors

- Treating the displacement current as moving charge.
- Using the static permittivity for light.
- Getting **B** the wrong way round; check that $\mathbf{E}\times\mathbf{B}$ points along the travel.
- Forgetting the factor $\frac{1}{2}$ in the average intensity.
- Saying the field at a superconductor's surface must be zero; only its normal part is.

Key terms

displacement current: $\varepsilon_0\, d\Phi_E/dt$, the term that completes Ampère's law.

Poynting vector: $\mathbf{E}\times\mathbf{B}/\mu_0$, the energy flow per unit area.

boundary conditions: Rules for how fields change across a surface between two media.

permittivity: How strongly a medium reduces an electric field; $\varepsilon = \varepsilon_r\varepsilon_0$.

intensity: Average power per unit area.

radiation pressure: The pressure light exerts because it carries momentum.

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Check yourself

- A non-magnetic medium has $\varepsilon_r = 4$. What are v and n?
- A wave has $E \propto \cos(kz + \omega t)$. Which way does it travel?
- Is the normal component of $\mathbf{B}$ continuous at a boundary?

Answers: $v = 1.5 \times 10^8$ m/s and $n = 2$. Towards $-z$. Yes, always.

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Acoustic Doppler echoes and standing-wave boundaries

GRE Subject Test · GRE Physics

Acoustic Doppler shifts and standing waves

In June 1845 the Dutch scientist Christophorus Buys Ballot tested an idea Christian Doppler had published three years before. He put trumpeters on an open railway carriage near Utrecht and had them play one steady note as the train passed musicians standing by the line. The listeners, who had perfect pitch, heard the note higher as the train came towards them and lower as it went away, by about the amount Doppler's formula predicted. This is the **Doppler effect** 多普勒效应.

GRE Physics questions on sound waves ask you to:

- Use the different formulas for a moving source and a moving observer.
- Handle echoes, which are shifted twice.
- Find standing-wave frequencies on strings and in pipes, with **nodes** 波节 and **antinodes** 波腹.
- Use **beats** 拍 to compare two frequencies.

Moving source or moving observer

Measure every speed relative to the air. With sound speed c:

- Source approaching a still observer at u_s: $f' = f\dfrac{c}{c - u_s}$.
- Observer approaching a still source at u_o: $f' = f\dfrac{c + u_o}{c}$.
- Receding changes the sign of the speed. The two formulas are not interchangeable: they give different shifts for the same speed.

Echoes from a fixed wall

An echo is shifted twice. The wall receives the moving source's shifted frequency and sends it back unchanged; the moving driver then hears it as a moving observer.

$$f_{\text{echo}} = f\,\frac{c + u}{c - u}$$

- For small u/c, the fractional shift is about $2u/c$. Police radar and ultrasound blood-flow meters use this.

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Standing waves on strings

A string fixed at both ends has nodes at the ends. Its allowed wavelengths are $2L/n$:

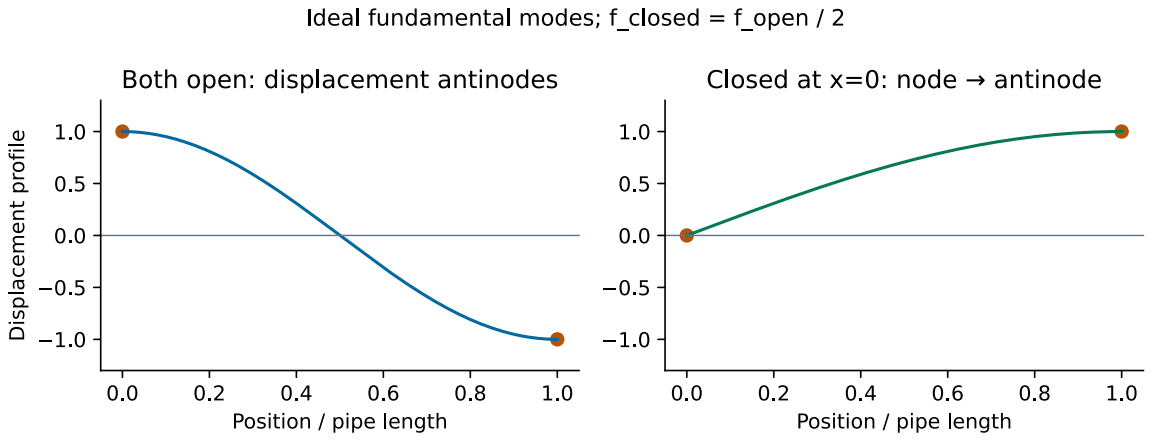
$$f_n = \frac{nv}{2L}, \qquad v = \sqrt{T/\mu}$$

- The lowest frequency is the **fundamental** 基频; the others are **harmonics** 谐波.
- Four times the tension doubles the wave speed and every frequency.

Pipes: nodes and antinodes

Air in a pipe has a displacement antinode at an open end and a node at a closed end.

- Both ends open: $f_n = \dfrac{nc}{2L}$, all whole numbers n.
- One end closed: $f_n = \dfrac{(2n-1)c}{4L}$, odd multiples of the fundamental only.



For the same length, the closed pipe's fundamental is half the open pipe's

Beats

Two notes of nearly equal frequency add to a sound whose loudness rises and falls. The beat frequency is the difference:

$$f_{\text{beat}} = |f_1 - f_2|$$

- Musicians tune by removing beats against a reference note.

Worked example: a siren and its echo

A driver approaches a wall at 10 m/s, sounding a 500 Hz siren; $c = 340$ m/s.

- Frequency reaching the wall: $500 \times 340/330 = 515$ Hz.
- Echo heard by the driver: $500 \times 350/330 = 530$ Hz.
- The driver hears the siren (500 Hz) and the echo (530 Hz) together: 30 beats per second, close to the estimate $2u/c \times 500 = 29$ Hz.

Worked example: closing one end of a pipe

A pipe is 0.85 m long; $c = 340$ m/s.

- Both ends open: $f_n = n(340)/(2 \times 0.85) = 200, 400, 600$ Hz, …
- One end closed: $f_n = (2n-1)(340)/(4 \times 0.85) = 100, 300, 500$ Hz, …
- The fundamental halves, and none of the old frequencies survives: they were all even multiples of $c/4L$.

Common errors

- Using the source formula when the observer moves, or the reverse.
- Measuring speeds relative to the ground when the air is moving (wind).
- Shifting an echo only once.
- Allowing even harmonics in a pipe closed at one end.
- Putting a node at an open end.

Key terms

Doppler effect: The change in observed frequency caused by motion of the source or observer.

node: A point of a standing wave that does not move.

antinode: A point of a standing wave with the largest motion.

beats: The regular rise and fall in loudness of two notes with nearly equal frequencies.

fundamental: The lowest frequency of a standing wave.

harmonics: The whole-number multiples of the fundamental frequency.

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Check yourself

- A 680 Hz source moves away from you at 20 m/s; $c = 340$ m/s. What do you hear?
- A string 0.65 m long carries waves at 260 m/s. What is its fundamental?
- Two tuning forks of 440 Hz and 444 Hz sound together. What is the beat frequency?

Answers: $680 \times 340/360 = 642$ Hz. $260/(2 \times 0.65) = 200$ Hz. 4 Hz.

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Optical instruments, path differences and refraction

GRE Subject Test · GRE Physics

In 1887 Albert Michelson and Edward Morley, in Cleveland, floated a heavy stone table on a pool of mercury so they could turn it smoothly. On it was an **interferometer** 干涉仪: a beam split into two arms at right angles, reflected back and forth to an effective length of about 11 m, then recombined. If light travelled through an "ether" that the Earth moved through, turning the table should have shifted the fringes by about 0.4 of a fringe. They saw less than a twentieth of that. Counting fringes is one of the most sensitive measurements in physics.

GRE Physics optics questions ask you to:

- Find the length and **angular magnification** 角放大率 of a telescope.
- Use the **resolving power** 分辨本领 of a grating.
- Count interferometer fringes, remembering the double pass.
- Use Snell's law, the **critical angle** 临界角 and **total internal reflection** 全反射.

The telescope

A refracting telescope has a long-focus objective and a short-focus eyepiece. For relaxed viewing (final image at infinity):

- The lenses are $f_o + f_e$ apart.
- The angular magnification is $|M| = f_o/f_e$, and the image is inverted.
- A larger objective collects more light and resolves finer detail; magnification alone does not.

Grating resolving power

A grating separates two wavelengths λ and $\lambda + \Delta\lambda$ if:

$$R = \frac{\lambda}{\Delta\lambda} = mN$$

- N is the number of slits lit and m the order. More slits make the peaks sharper; the slit spacing only moves them.
- The sodium lines at 589.0 nm and 589.6 nm need $R \approx 1000$.

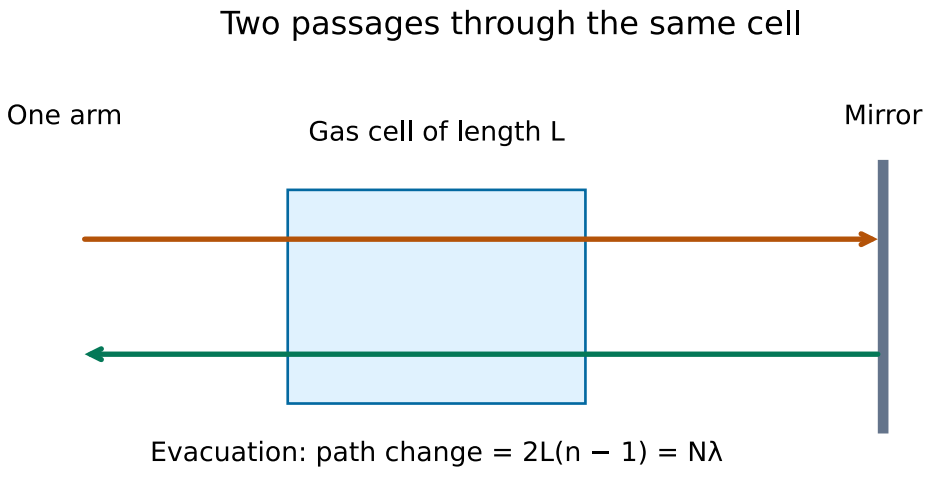
Counting fringes

In a Michelson interferometer, each fringe that passes means one wavelength of change in optical path. Light crosses each arm twice.

- Moving a mirror by Δx changes the path by $2\Delta x$: $N = 2\Delta x/\lambda$.

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- A gas cell of length L in one arm, emptied or filled, changes the path by $2L(n-1)$.



Evacuating the cell changes the optical path by $2L(n-1) = N\lambda$

Snell's law

At a boundary, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where the **refractive index** 折射率 is $n = c/v$ and the angles are measured from the normal.

- Use the normal of the surface the ray actually crosses; adjacent faces of a block have perpendicular normals.
- Light bends towards the normal when it slows down.

Total internal reflection

Going from a denser medium to a less dense one, the refracted ray grazes the surface at the critical angle, $\sin\theta_c = n_2/n_1$. Beyond it, all the light reflects.

- Glass ($n = 1.5$) into air: $\theta_c = 41.8°$. Water ($n = 1.33$): $48.8°$.
- Optical fibres and prism binoculars use total internal reflection.

Worked example: a gas cell

A 5.0 cm gas cell sits in one arm of a Michelson interferometer lit at 500 nm. Pumping out the gas moves 80 fringes past.

- Path change: $2L(n-1) = N\lambda$.
- $n - 1 = \dfrac{80 \times 500 \times 10^{-9}}{2 \times 0.050} = 4.0 \times 10^{-4}$.
- Forgetting the double pass would give 8.0×10^{-4}, twice too large.

Worked example: telescope and grating

A telescope is 80 cm long in relaxed viewing, with an eyepiece of $f_e = 16$ cm.

- $f_o = 80 - 16 = 64$ cm, so $|M| = 64/16 = 4$.
- To resolve the sodium lines in second order: $R = 589.0/0.6 = 982$, so $N \geq 982/2 = 491$ slits must be lit.
- A 600 nm pair 3 nm apart needs only $R = 200$.

Common errors

- Taking the telescope length as $f_o - f_e$ or the magnification as f_e/f_o.
- Thinking finer slit spacing alone improves resolving power.
- Forgetting the factor of 2 for the return trip in an interferometer.
- Measuring angles from the surface instead of the normal.
- Expecting total internal reflection going from air into glass.

Key terms

interferometer: An instrument that splits light and recombines it to measure path differences.

angular magnification: The ratio of the angle an image fills to the angle the object fills.

resolving power: $\lambda/\Delta\lambda$, a measure of how close two wavelengths can be and still be separated.

critical angle: The angle of incidence beyond which total internal reflection happens.

total internal reflection: Complete reflection of light at a boundary with a less dense medium.

refractive index: The ratio of the speed of light in a vacuum to its speed in a medium.

Check yourself

- A Michelson mirror moves 0.10 mm with 500 nm light. How many fringes pass?
- What is the critical angle from water ($n = 1.33$) into air?
- What resolving power separates 600 nm lines 3 nm apart?

Answers: $2(1.0 \times 10^{-4})/(5.0 \times 10^{-7}) = 400$. $\sin^{-1}(1/1.33) = 48.8°$. $R = 200$.

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Fourier symmetry and wave superposition

GRE Subject Test · GRE Physics

In 1807 Joseph Fourier sent the Paris Academy a long paper on how heat spreads through a solid. To solve his equations he claimed that any function, even one with corners or jumps, could be written as a sum of sines and cosines. Joseph-Louis Lagrange and others doubted it, and the paper was not published for years. Today the **Fourier series** 傅里叶级数 is one of the most used tools in physics, from heat flow to music to quantum mechanics.

GRE Physics questions on Fourier analysis and superposition ask you to:

- Find Fourier **coefficients** 系数 and the mean value.
- Use symmetry (even and odd functions) to kill terms before integrating.
- Connect smoothness with how fast the coefficients fall.
- Add coherent waves by amplitude.

Fourier coefficients

A function with period 2π can be written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

- $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.
- The mean value of f is $a_0/2$, not a_0.
- The terms with $n > 1$ are the **harmonics** 谐波 of the fundamental frequency.

Symmetry kills terms

Symmetry often tells you which coefficients vanish without any integration.

- An **even function** 偶函数, $f(-x) = f(x)$, has only cosines: every $b_n = 0$.
- An **odd function** 奇函数, $f(-x) = -f(x)$, has only sines: the mean and every a_n are zero.
- Symmetry depends on where you put the origin. Shifting a signal can mix sines and cosines without changing which frequencies it contains.

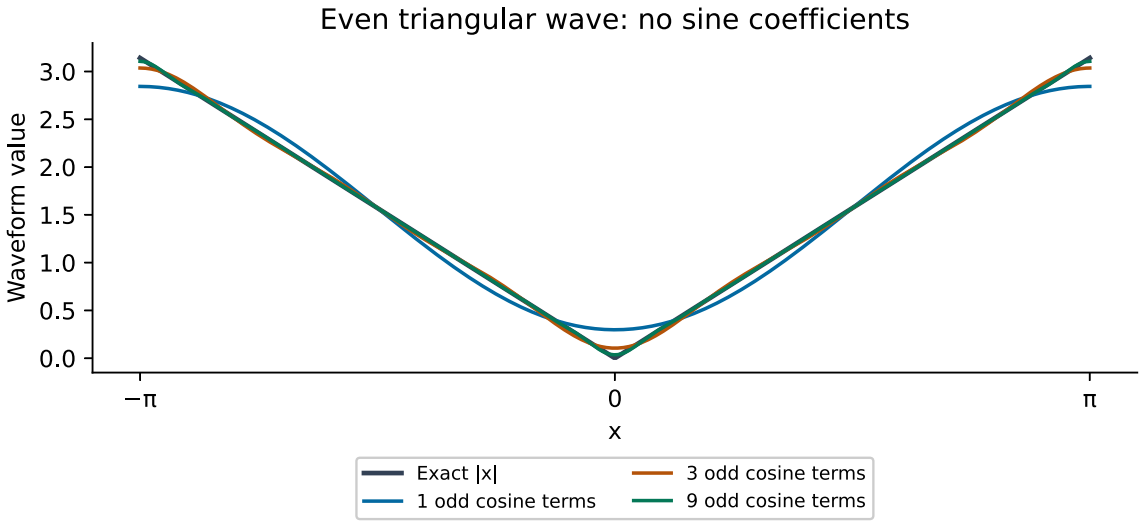
The triangle wave

$f(x) = |x|$ on $[-\pi, \pi]$, repeated, is even, so it has only cosines. Its mean is $\pi/2$ and:

$$a_n = \frac{2\left[(-1)^n - 1\right]}{\pi n^2} = -\frac{4}{\pi n^2} \text{ for odd } n, \quad 0 \text{ for even } n$$

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- The coefficients fall as $1/n^2$, because the function is continuous but has corners.



An even function needs only cosine terms; more terms sharpen the corner

The square wave and the Gibbs overshoot

A square wave, $+1$ for $0 < x < \pi$ and -1 for $-\pi < x < 0$, is odd. Its coefficients are $b_n = 4/(\pi n)$ for odd n and zero for even n.

- A jump makes the coefficients fall only as $1/n$, more slowly than for the triangle wave.
- Near each jump, the partial sums overshoot by about 9% of the jump, however many terms you add: the **Gibbs phenomenon** 吉布斯现象.
- Smoother functions need fewer terms: a corner gives $1/n^2$, a jump gives $1/n$.

Coherent superposition

When waves are coherent, add their amplitudes, then square. Two equal waves with phase difference φ give:

$$I = 2I_0(1 + \cos\varphi)$$

- At $\varphi = \pi/2$, $I = 2I_0$; at $\varphi = \pi$, zero.
- For incoherent light the cross term averages to zero, so the intensity is always $2I_0$.

Worked example: partial sums of the triangle wave

Use the series for $|x|$ at $x = 0$, where the true value is 0.

- One term: $\dfrac{\pi}{2} - \dfrac{4}{\pi}\cos 0 = 1.571 - 1.273 = 0.298$.
- Two odd terms: $1.571 - 1.273\left(1 + \frac{1}{9}\right) = 0.156$.
- The error falls slowly towards zero as more odd terms are added, as the $1/n^2$ coefficients predict.

Worked example: the square wave at its peak

For the square wave above, evaluate the first three non-zero terms at $x = \pi/2$, where the true value is 1.

- $\dfrac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2}\right) = 1.273\left(1 - \frac{1}{3} + \frac{1}{5}\right)$.
- $= 1.273 \times 0.867 = 1.10$, already close to 1.
- The first term alone has amplitude $4/\pi = 1.27$, larger than the square wave itself.

Common errors

- Taking the mean as a_0 instead of $a_0/2$.
- Expecting sine terms in an even function.
- Forgetting that symmetry depends on the chosen origin.
- Expecting the Gibbs overshoot to vanish with more terms.
- Adding intensities for coherent waves.

Key terms

Fourier series: A sum of sines and cosines that represents a periodic function.

coefficient: The number that multiplies one term of a series.

harmonics: Frequencies that are whole-number multiples of the fundamental.

even function: A function with $f(-x) = f(x)$.

odd function: A function with $f(-x) = -f(x)$.

Gibbs phenomenon: The overshoot of a Fourier sum near a jump, which does not shrink as terms are added.

Check yourself

- Which terms appear in the Fourier series of x^3 on $[-\pi, \pi]$?
- A Fourier series has $a_0 = 4$. What is the mean of the function?
- Whose coefficients fall faster, the triangle wave's or the square wave's?

Answers: sines only, because x^3 is odd. 2. The triangle wave's ($1/n^2$ against $1/n$).

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Gas processes, entropy and reversible cycles

GRE Subject Test · GRE Physics

In 1852 William Thomson, later Lord Kelvin, published a short paper with a surprising idea. A machine could warm a building by pumping heat in from the colder air outside, and it would need less fuel than burning that fuel in a stove. He called it a "heat multiplier". Today we call it a **heat pump** 热泵. It does not break any law: it moves heat rather than creating it, and its **coefficient of performance** 性能系数 can be well above 1.

GRE Physics thermodynamics questions ask you to:

- Apply the first law along different paths, with clear signs.
- Read work and heat from P–V and T–S diagrams.
- Treat entropy as a **state function** 状态函数.
- Use the Carnot limits for engines, refrigerators and heat pumps.

The first law along a path

$\Delta U = Q - W$, with $W = \int P\,dV$ the work done by the gas. For an ideal gas, U depends only on temperature.

- Constant volume: $W = 0$, so $\Delta U = Q$.
- **Isothermal** 等温的: $\Delta U = 0$, so $Q = W$.
- **Adiabatic** 绝热的: $Q = 0$, so $\Delta U = -W$. The temperature changes.
- Round a complete **cycle** 循环: $\Delta U = 0$, so the net heat in equals the net work out.

Reading P–V diagrams

Work is the area under the path on a P–V diagram. Use pascals and cubic metres.

- An isotherm is $PV =$ constant; an adiabat, $PV^{\gamma} =$ constant, is steeper.
- A cycle run clockwise does positive net work, equal to the enclosed area.
- A rectangle from 1 to 3×10^5 Pa and 0.001 to 0.004 m^3 encloses $2\times10^5\times0.003 = 600$ J.

Adiabatic processes

With no heat exchanged, a gas cools as it expands and warms as it is compressed:

$$PV^{\gamma} = \text{constant}, \qquad TV^{\gamma-1} = \text{constant}$$

- $\gamma = 5/3$ for a monatomic gas and $7/5$ for air (diatomic) near room temperature.

- A diesel engine ignites its fuel by adiabatic compression alone.

Entropy

$dS = \delta Q_{\text{rev}}/T$. Because entropy is a state function, the change between two states is the same for any path; calculate it along a reversible one.

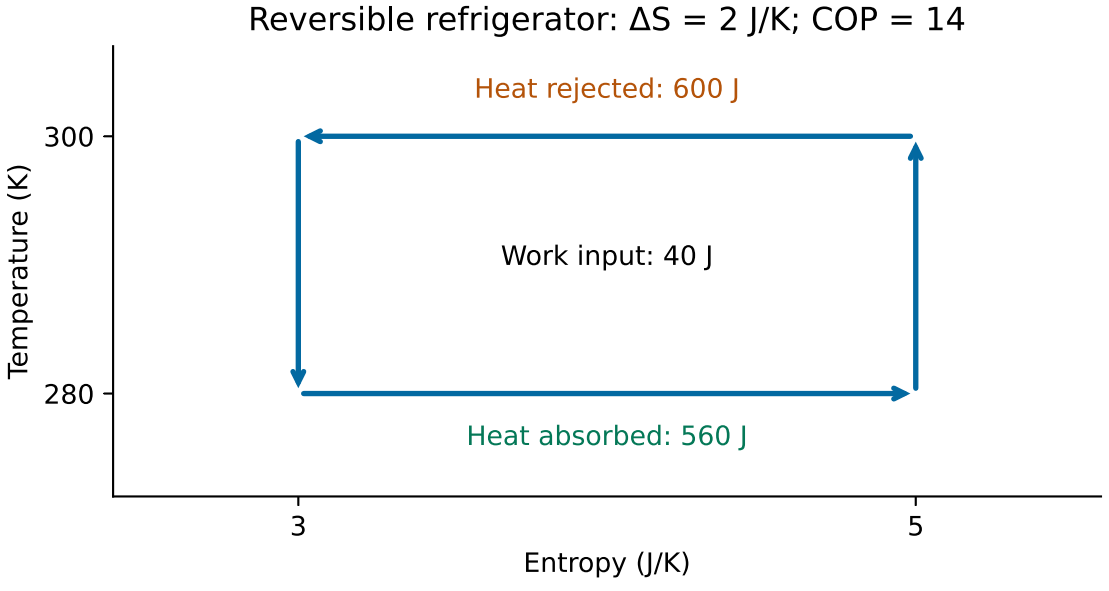
- Reversible isothermal expansion: $\Delta S_{\text{gas}} = nR\ln(V_2/V_1)$; the reservoir loses the same amount.
- A reversible process has zero total entropy change, though the system's entropy can change.
- On a T–S diagram, the area under a reversible path is the heat taken in.

Engines, refrigerators and heat pumps

The Carnot limits, with temperatures in kelvin:

$$\eta = 1 - \frac{T_C}{T_H}, \qquad \text{COP}_{\text{fridge}} = \frac{T_C}{T_H - T_C}, \qquad \text{COP}_{\text{pump}} = \frac{T_H}{T_H - T_C}$$

- For the same temperatures, $\text{COP}_{\text{pump}} = \text{COP}_{\text{fridge}} + 1$, because the heat delivered includes the work.
- Both COPs grow as the temperatures get closer.



For a reversible refrigerator, COP = 560/40 = 14

Worked example: a Carnot refrigerator

A reversible refrigerator works between 280 K and 300 K and moves 2.0 J/K of entropy each cycle.

- Heat taken from inside: $Q_C = T_C \Delta S = 280 \times 2.0 = 560$ J. Heat rejected: $Q_H = 300 \times 2.0 = 600$ J.
- Work: $W = 600 - 560 = 40$ J, so COP $= 560/40 = 14 = 280/20$.
- As a heat pump between the same temperatures, it would deliver 600 J for 40 J of work: COP 15.

Worked example: compressing air

Air ($\gamma = 1.4$) at 300 K is compressed adiabatically to half its volume.

- $T_2 = T_1(V_1/V_2)^{\gamma-1} = 300 \times 2^{0.4} = 396$ K.
- $P_2/P_1 = 2^{1.4} = 2.64$, more than the factor 2 an isothermal compression would give.
- The work done on the gas becomes internal energy, which is why a bicycle pump gets warm.

Common errors

- Thinking "adiabatic" means constant temperature.
- Using litres and kilopascals in $W = P\Delta V$ without converting.
- Calling a COP of 14 an efficiency of 1400%.
- Saying a reversible process leaves the system's entropy unchanged; only the total is unchanged.
- Using degrees Celsius in the Carnot formulas.

Key terms

heat pump: A device that uses work to move heat from a cold place to a warm one, in order to heat the warm one.

coefficient of performance: Useful heat moved divided by the work supplied.

state function: A quantity that depends only on the present state, not on how it was reached.

isothermal: At constant temperature.

adiabatic: With no heat exchanged.

cycle: A series of processes that returns a system to its starting state.

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Check yourself

- A rectangular cycle runs between 1×10^5 and 2×10^5 Pa and between 2 and 5 litres. What is the net work?
- In an adiabatic expansion, does the temperature of an ideal gas change?
- What is the largest COP of a heat pump between 270 K and 300 K?

Answers: $(1 \times 10^5)(3 \times 10^{-3}) = 300$ J. Yes, it falls. $300/30 = 10$.

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Speed distributions and statistical ensembles

GRE Subject Test · GRE Physics

In 1920 Otto Stern, in Frankfurt, heated silver in an oven and let a thin beam of silver atoms escape into a vacuum. The beam passed through a slit in a fast-spinning cylinder and landed on a glass plate. Because the cylinder turned while the atoms flew across it, slower atoms landed further round. The spread of silver on the plate showed the speeds directly, and they agreed roughly with the distribution James Clerk Maxwell had derived in 1860. A gas has no single speed, only a **probability density** 概率密度 of speeds.

GRE Physics statistical mechanics questions ask you to:

- Normalise and read probability densities.
- Use the **Maxwell distribution** 麦克斯韦分布 and its three typical speeds.
- Apply **equipartition** 能量均分.
- Find canonical probabilities with **degeneracy** 简并度.

Velocity is not speed

In still air the molecules' average velocity is zero, because every direction is equally likely. Their average speed is not zero: about 470 m/s for nitrogen at room temperature.

- Opposite velocities cancel in an average; speeds, which are always positive, do not.

Densities, not probabilities

$f(v)$ is a density: $f(v)\,dv$ is the probability of a speed between v and $v + dv$.

- f has units of 1/speed, and $\int_0^\infty f(v)\,dv = 1$. A count density integrates to N instead.
- For a continuous distribution, the probability of one exact value is zero; only intervals carry probability.

The Maxwell distribution

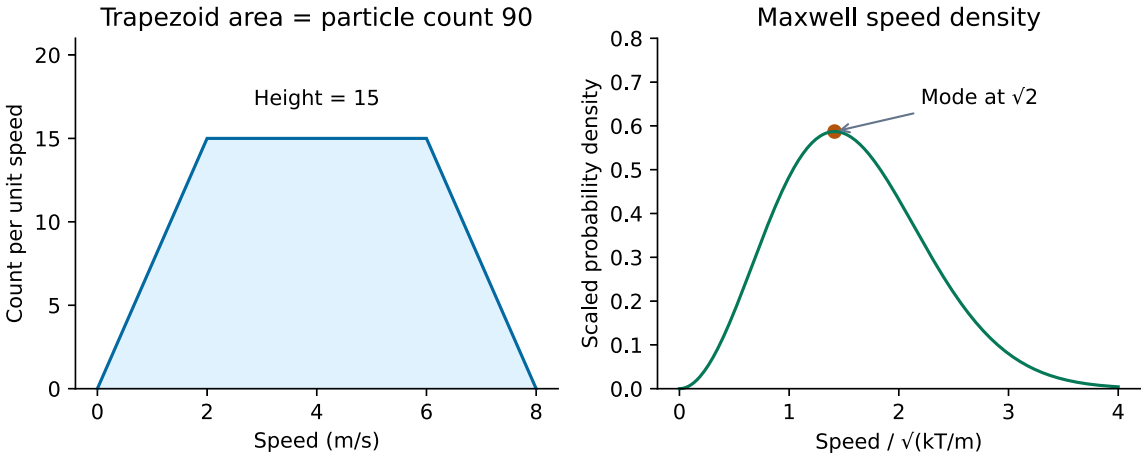
For an ideal gas in equilibrium:

$$f(v) \propto v^2\, e^{-mv^2/2kT}$$

- Most probable speed $\sqrt{2kT/m}$ < mean speed $\sqrt{8kT/\pi m}$ < **root-mean-square speed** 方均根速率 $\sqrt{3kT/m}$.

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- $f(0) = 0$ because of the v^2 factor, which counts the shell of velocity vectors with speed v.
- All speeds scale as $\sqrt{T/m}$: four times the temperature doubles them.



The area under a distribution counts the particles; the Maxwell mode is √(2kT/m)

Equipartition

In classical equilibrium, each quadratic term in the energy has an average of $\frac{1}{2}kT$.

- A molecule's translational kinetic energy has three terms, so its average is $\frac{3}{2}kT$.
- A diatomic molecule adds two rotational terms: $\frac{5}{2}kT$ in total, so $C_V = \frac{5}{2}R$ per mole.
- Quantum effects freeze out terms whose energy steps are large compared with kT (vibrations at room temperature).

Canonical probabilities with degeneracy

In the **canonical ensemble** 正则系综 (a system at fixed temperature), every state has the weight $e^{-E/kT}$. A level with g states gets g times that weight.

$$P(\text{level } i) = \frac{g_i\, e^{-E_i/kT}}{Z}$$

- Each state counts equally, not each level.
- Ground level $g = 1$ and excited level $g = 2$ at a gap of $kT \ln 2$: weights 1 and $2 \times \frac{1}{2} = 1$. Each level has probability $\frac{1}{2}$, and each excited state $\frac{1}{4}$.

Worked example: normalising a trapezoid

A count density rises linearly from 0 over a speed range v_0, stays flat at height a for $2v_0$, then falls to 0 over v_0. It describes $N = 90$ particles with $v_0 = 2.0$ m/s.

- Area: $\frac{1}{2}av_0 + 2av_0 + \frac{1}{2}av_0 = 3av_0$.
- $3a(2.0) = 90$, so $a = 15$ particles per (m/s).
- Particles on the flat part: $15 \times 4.0 = 60$.

Worked example: nitrogen at 300 K

A nitrogen molecule has $m = 28 \times 1.66 \times 10^{-27} = 4.65 \times 10^{-26}$ kg.

- rms speed: $\sqrt{3(1.38 \times 10^{-23})(300)/(4.65 \times 10^{-26})} = 517$ m/s.
- Mean speed: $\sqrt{8/3\pi} \times 517 = 476$ m/s. Most probable: $\sqrt{2/3} \times 517 = 422$ m/s.
- At 1200 K, all three double.

Common errors

- Reading a density value as a probability.
- Giving a non-zero probability to one exact speed.
- Putting the peak of the Maxwell speed distribution at $v = 0$.
- Mixing up the order of the mode, mean and rms speeds.
- Forgetting the degeneracy g_i before dividing by Z.

Key terms

probability density: A function whose area over an interval gives the probability of that interval.

Maxwell distribution: The probability density of molecular speeds in an ideal gas in equilibrium.

root-mean-square speed: The square root of the average of v^2.

equipartition: Each quadratic energy term has an average of $\frac{1}{2}kT$ in classical equilibrium.

canonical ensemble: All the states of a system in contact with a heat bath at fixed temperature.

degeneracy: The number of states that share one energy level.

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Check yourself

- The temperature of a gas is quadrupled. What happens to its rms speed?
- What is the probability that a molecule has a speed of exactly 400 m/s?
- A ground level ($g = 1$) and an excited level ($g = 3$) are $kT \ln 3$ apart. What is the probability of each level?

Answers: it doubles. Zero. $\frac{1}{2}$ each, since the weights are 1 and $3 \times \frac{1}{3} = 1$.

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Counting statistics, uncertainty and dimensional models

GRE Subject Test · GRE Physics

In 1910 Ernest Rutherford and Hans Geiger counted the tiny flashes made by alpha particles from a radioactive source hitting a zinc sulphide screen. They counted in 2608 intervals of 7.5 seconds each, averaging 3.87 flashes per interval. Some intervals had none and some had ten or more. Harry Bateman showed that the spread matched one formula exactly: the **Poisson distribution** 泊松分布. Random events are never perfectly regular, and the spread itself can be predicted.

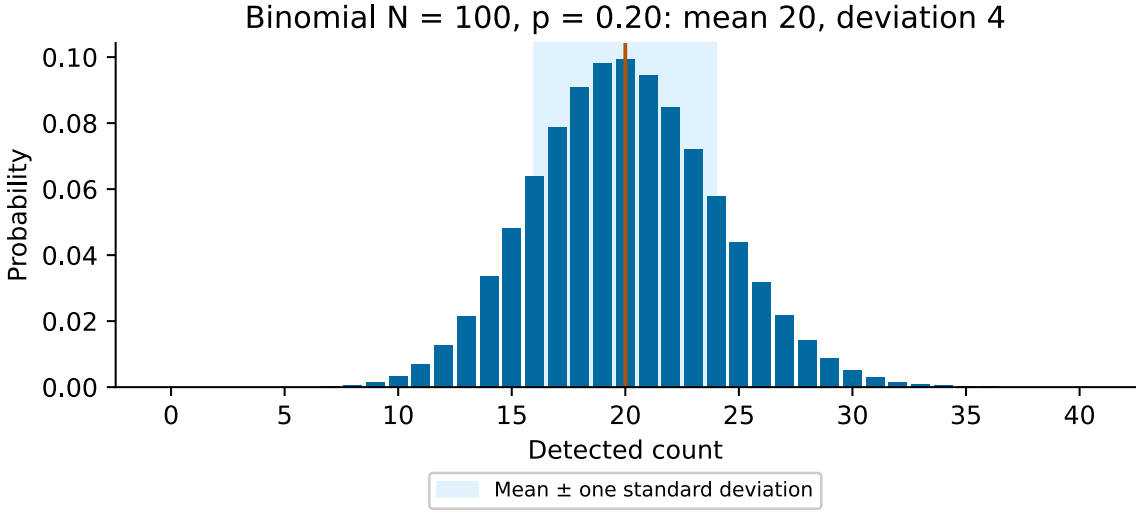
GRE Physics laboratory questions ask you to:

- Use the binomial and Poisson distributions and their **standard deviation** 标准差.
- Propagate uncertainties through formulas.
- Separate random error from bias, and use the **standard error** 标准误差 of a mean.
- Use **dimensional analysis** 量纲分析 to find how a quantity depends on others.

The binomial distribution

N independent trials, each with probability p of success, follow the **binomial distribution** 二项分布.

- Mean Np, **variance** 方差 $Np(1-p)$, standard deviation $\sqrt{Np(1-p)}$.
- A detector with 20% efficiency that receives 100 photons records 20 ± 4, not exactly 20.



Mean $Np = 20$, standard deviation $\sqrt{(Np(1-p))} = 4$

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The Poisson distribution

When N is large and p is small, the binomial becomes Poisson, with mean $\lambda = Np$:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad \sigma = \sqrt{\lambda}$$

- Counts of radioactive decays, photons or cosmic rays are Poisson: N counts have an uncertainty of $\sqrt{N}$.
- $N = 200$, $p = 0.05$: exact $\sigma = \sqrt{9.5} = 3.08$; Poisson $\sqrt{10} = 3.16$, close because p is small.

Propagating uncertainty

For independent inputs, the uncertainties combine in quadrature:

$$\sigma_y^2 \approx \sum_i \left(\frac{\partial y}{\partial x_i}\right)^2 \sigma_i^2$$

- For $y = x^a$, the relative uncertainty of y is $|a|$ times that of x.
- For products and quotients, add relative uncertainties in quadrature. Adding them in a straight line gives only a worst case.

Precision, accuracy and averaging

Averaging n independent readings shrinks the random uncertainty of the mean, its standard error, by $1/\sqrt{n}$.

- A bias shared by every reading (a badly calibrated meter) is not reduced by averaging.
- To halve the standard error, you need four times as many readings.

Dimensional analysis

Write each quantity in powers of mass (M), length (L) and time (T), and match the powers on both sides.

- $[G] = \mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}$, $[\hbar] = \mathrm{ML}^2\mathrm{T}^{-1}$, $[c] = \mathrm{LT}^{-1}$.
- A pendulum's period can depend only on L and g: $T \propto \sqrt{L/g}$, and mass cannot enter.
- Dimensions fix the powers but not pure numbers such as 2π.

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Worked example: counting uncertainties

A counter records 400 decays in a minute; the speed in a formula $K = \frac{1}{2}mv^2$ is known to 4%.

- Counts: $400 \pm \sqrt{400} = 400 \pm 20$, a 5% uncertainty. For 1%, you need $1/\sqrt{N} = 0.01$: 10,000 counts.
- Kinetic energy with exact mass: $2 \times 4\% = 8\%$.
- With 3% in m and 2% in v: $\sqrt{3^2 + (2 \times 2)^2} = 5\%$.

Worked example: the Planck length

Find a length from G, $\hbar$ and c: $\ell = G^a \hbar^b c^d$.

- M: $-a + b = 0$. L: $3a + 2b + d = 1$. T: $-2a - b - d = 0$.
- So $a = b = \frac{1}{2}$ and $d = -\frac{3}{2}$: $\ell = \sqrt{G\hbar/c^3}$.
- $\ell = \sqrt{(6.67 \times 10^{-11})(1.05 \times 10^{-34})/(3.0 \times 10^8)^3} = 1.6 \times 10^{-35}$ m.

Common errors

- Expecting a detector of 20% efficiency to record exactly 20% of events.
- Using the Poisson $\sqrt{\lambda}$ when p is not small.
- Adding independent uncertainties linearly.
- Forgetting to multiply the relative uncertainty by the power.
- Thinking more readings remove a calibration bias.

Key terms

Poisson distribution: The distribution of counts of independent random events, with variance equal to its mean.

standard deviation: The typical spread of values about the mean; the square root of the variance.

standard error: The uncertainty of a mean, $\sigma/\sqrt{n}$ for n independent readings.

dimensional analysis: Matching the powers of mass, length and time on both sides of an equation.

binomial distribution: The distribution of the number of successes in N independent trials.

variance: The average of the squared differences from the mean.

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Check yourself

- A counter records 900 counts. What is the uncertainty?
- A quantity depends on x^3, and x has a 2% uncertainty. What is the uncertainty of the quantity?
- How many readings are needed to make the standard error a third of that from one reading?

Answers: $\sqrt{900} = 30$. 6%. Nine.

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Relativistic lifetime, energy and Doppler shift

GRE Subject Test · GRE Physics

In 1963 David Frisch and James Smith counted cosmic-ray muons near the top of Mount Washington in New Hampshire, about 1900 m above sea level, then counted again at sea level in Cambridge, Massachusetts. A muon at rest lives only 2.2 μs on average. Even at almost the speed of light, it would need about 6.4 μs to fall 1900 m, so without relativity only about 30 of every 560 muons should have survived. They found about 410 at sea level. The muons' clocks were running slow: **time dilation** 时间膨胀.

GRE Physics relativity questions ask you to:

- Use **proper time** 固有时, time dilation and **length contraction** 长度收缩.
- Find a rest mass from energy and momentum with the **invariant** 不变量 $E^2-(pc)^2$.
- Compute relativistic kinetic energy.
- Use the relativistic Doppler shift and velocity addition.

Proper time and dilation

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}$$

- A clock moving with a particle measures its proper time τ_0; the lab measures $\gamma\tau_0$.
- Seen from the muon, the mountain is contracted by γ instead: the two views agree on how many muons survive.
- Decay is random, so these are mean lifetimes and mean distances.

The invariant mass

Energy and momentum change from frame to frame, but one combination does not:

$$E^2 - (pc)^2 = (mc^2)^2$$

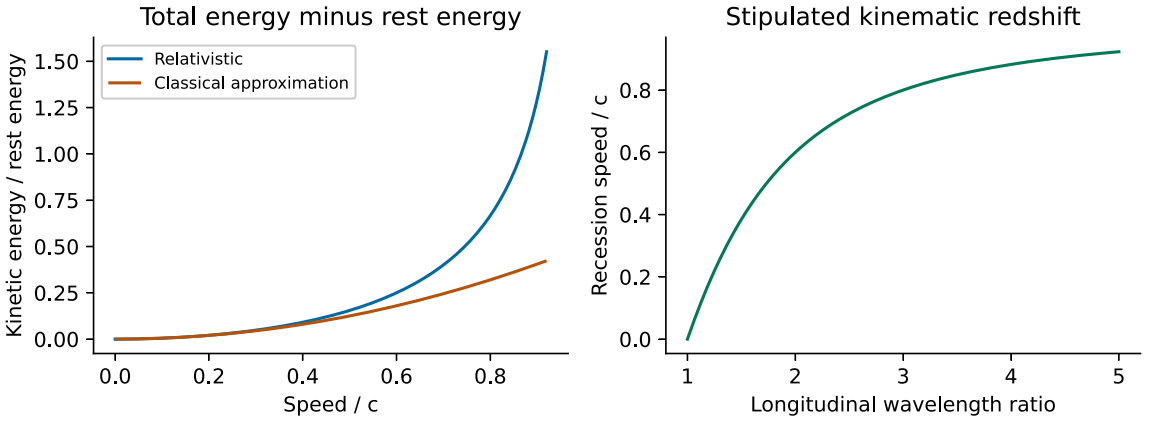
- So $mc^2 = \sqrt{E^2-(pc)^2}$, never $E - pc$.
- For a system of particles, use the total energy and total momentum: this gives the **invariant mass** 不变质量 of the system.
- Massless particles have $E = pc$.

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Kinetic energy

$K = (\gamma - 1)mc^2$. At $\beta = 0.6$, $\gamma = 1.25$ and $K = 0.25\,mc^2$, while the classical $\frac{1}{2}mv^2$ gives only $0.18\,mc^2$.

- Work raises γ without limit, so a massive particle never reaches c.



Classical ½mv² underestimates kinetic energy at high speed; a wavelength ratio of 2 means 0.6c

Relativistic Doppler shift

For a source moving straight away, the wavelength ratio is:

$$r = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \sqrt{\frac{1+\beta}{1-\beta}}, \qquad \beta = \frac{r^2 - 1}{r^2 + 1}$$

- A longer observed wavelength is a **redshift** 红移; an approaching source gives $r < 1$.
- The low-speed rule $\beta \approx r - 1$ works only for small shifts.
- Cosmological redshifts of distant galaxies need a different model, based on the expansion of space.

Velocity addition

Speeds do not simply add. A body moving at u in a frame that moves at v has, in the lab:

$$u' = \frac{u + v}{1 + uv/c^2}$$

- The result never exceeds c. Light moves at c in every frame.

Worked example: a fast particle's range

A particle with a proper mean lifetime of 3.0 μs moves at $0.80c$.

- $\gamma = 1/\sqrt{1 - 0.64} = 5/3$, so its lab lifetime is 5.0 μs.
- Mean distance: $0.80 \times (3.0 \times 10^{8}) \times (5.0 \times 10^{-6}) = 1200$ m.
- Without dilation: $0.80 \times (3.0 \times 10^{8}) \times (3.0 \times 10^{-6}) = 720$ m.

Worked example: masses and Doppler

- A particle has $E = 13$ GeV and $pc = 12$ GeV: $mc^2 = \sqrt{169 - 144} = 5$ GeV, not 1 GeV.
- Two 1.0 GeV photons moving in opposite directions: total momentum zero, so the system's invariant mass is 2.0 GeV/c^2. Moving the same way, it is zero.
- A line arrives at twice its emitted wavelength: $\beta = (4 - 1)/(4 + 1) = 0.6$.

Common errors

- Taking the rest energy as $E - pc$.
- Using $\frac{1}{2}mv^2$ at high speed.
- Applying dilation the wrong way: the proper lifetime is the shortest.
- Adding the masses of particles instead of finding the system's invariant mass.
- Adding velocities simply, giving speeds above c.

Key terms

time dilation: A moving clock runs slow by the factor γ.

proper time: The time measured by a clock moving with the object.

length contraction: A moving object is shorter along its motion by the factor γ.

invariant: A quantity with the same value in every inertial frame.

invariant mass: $\sqrt{E^2 - (pc)^2}/c^2$ for a particle or a system of particles.

redshift: An increase in the observed wavelength.

Check yourself

- A particle has $E = 5$ GeV and $pc = 4$ GeV. What is mc^2?
- A spectral line arrives at three times its emitted wavelength. What is β?
- A rocket at $0.6c$ fires a probe forward at $0.6c$ relative to itself. How fast is the probe in the lab?

Answers: 3 GeV. $\beta = (9 - 1)/(9 + 1) = 0.8$. $1.2c/1.36 = 0.88c$.

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Photoelectrons, reduced mass and atomic excitation

GRE Subject Test · GRE Physics

In 1914 James Franck and Gustav Hertz sent electrons through mercury vapour and slowly raised the accelerating voltage. The current rose, then dropped sharply at 4.9 V, rose again and dropped at 9.8 V, then at 14.7 V. Each drop meant that electrons had just enough energy to lose exactly 4.9 eV in a collision with a mercury atom, once, twice or three times. The excited atoms then gave out ultraviolet light at 254 nm, which is 1240/4.9. Atoms absorb energy only in fixed amounts.

GRE Physics questions on atoms and photons ask you to:

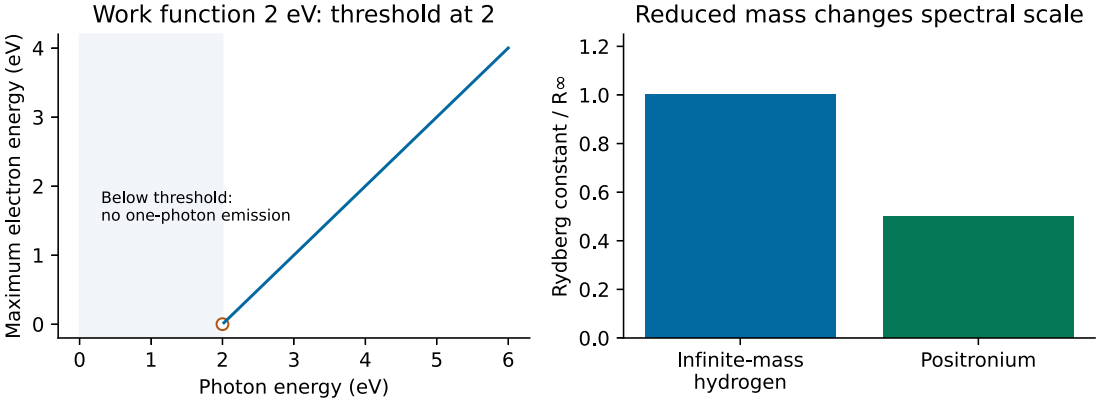
- Use the photoelectric equation with the **work function** 逸出功 and **stopping potential** 遏止电压.
- Read X-ray spectra: **characteristic X-rays** 特征 X 射线 and **bremsstrahlung** 韧致辐射.
- Use **reduced mass** 约化质量 for two-body atoms.
- Interpret Franck–Hertz spacings.

The photoelectric effect

A photon of energy $h\nu$ can free one electron from a metal. The fastest electrons have:

$$K_{\max} = h\nu - \phi, \qquad eV_s = K_{\max}$$

- Below the **threshold frequency** 截止频率 $\nu_0 = \phi/h$, no electrons come out, however bright the light.
- Brighter light at a fixed frequency gives more electrons (more current), not faster ones.
- In 1916 Robert Millikan plotted V_s against ν and found a straight line of slope h/e.



No emission below 2 eV; positronium's reduced mass halves its spectral scale

X-ray spectra

An X-ray tube fires electrons at a metal target. Its spectrum has two parts.

- Characteristic lines: an electron knocks out an inner electron, and an outer one falls into the gap. The lines are sharp and set by the target. For the K_α line, $E \approx 10.2\,\text{eV} \times (Z-1)^2$ (Moseley's law).
- Bremsstrahlung: electrons slowing in the target give a continuous spectrum. It ends at the tube energy eV, so $\lambda_{\min} = hc/eV$.

Reduced mass

In a two-body atom, both particles move round their centre of mass. Replace the electron mass by the reduced mass:

$$\mu = \frac{m_e M}{m_e + M}$$

- Energy levels scale as μZ^2. Deuterium's lines are slightly shorter in wavelength than hydrogen's.
- Positronium (an electron and a positron): $\mu = m_e/2$, so its energies are halved and its wavelengths doubled.

Spin pairing and Hund's rule

Electrons fill degenerate orbitals singly, with parallel spins, before any pair up.

- Carbon ($2p^2$) and oxygen ($2p^4$) both have two unpaired electrons and total spin $S = 1$.
- Nitrogen ($2p^3$) has three unpaired electrons: $S = 3/2$.

Franck–Hertz spacings

A spacing ΔV between current dips means an energy $e\Delta V$ lost per collision.

- Use the spacing, not the position of the first dip: contact potentials shift every dip by the same amount.
- The excited atom returns to the ground state by emitting a photon of $\lambda = 1240/(e\Delta V \text{ in eV})$ nm.

Worked example: a photocell

Light of 8.0×10^{14} Hz falls on a metal with $\phi = 2.0$ eV; $h = 4.14 \times 10^{-15}$ eV s.

- Photon energy: $h\nu = 3.31$ eV.
- $K_{\max} = 3.31 - 2.0 = 1.31$ eV, so $V_s = 1.31$ V.
- Threshold: $\nu_0 = 2.0/(4.14 \times 10^{-15}) = 4.8 \times 10^{14}$ Hz.

Worked example: X-rays from copper

A copper target ($Z = 29$) sits in a 30 kV tube.

- K_α: $E \approx 10.2 \times 28^2 = 8.0$ keV, close to the measured 8.05 keV.
- Shortest bremsstrahlung wavelength: $\lambda_{\min} = 1240/30{,}000 = 0.041$ nm.
- Positronium's Lyman-alpha line is twice hydrogen's 122 nm: 243 nm.

Common errors

- Thinking brighter light gives faster photoelectrons.
- Reading the energy from the first Franck–Hertz dip instead of the spacing.
- Confusing characteristic lines with the bremsstrahlung continuum.
- Using the electron mass for positronium.
- Pairing electrons before every degenerate orbital has one.

Key terms

work function: The least energy needed to free an electron from a metal surface.

stopping potential: The reverse voltage that just stops the fastest photoelectrons.

threshold frequency: The lowest light frequency that frees electrons.

characteristic X-rays: Sharp X-ray lines from inner-shell transitions, set by the element.

bremsstrahlung: Radiation from charges that slow down; a continuous X-ray spectrum.

reduced mass: $m_1 m_2/(m_1 + m_2)$, the effective mass in a two-body problem.

Check yourself

- The light on a photocell doubles in intensity at the same frequency. What happens to $K_{\max}$ and the current?
- What is the shortest X-ray wavelength from a 50 kV tube?
- Franck–Hertz dips are 4.9 V apart. What wavelength do the excited atoms emit?

Answers: $K_{\max}$ is unchanged; the current doubles. $1240/50{,}000 = 0.025$ nm. $1240/4.9 = 253$ nm.

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Scattering, degeneracy and Pauli operators

GRE Subject Test · GRE Physics

In February 1922 Otto Stern and Walther Gerlach, in Frankfurt, sent a beam of silver atoms through a strongly non-uniform magnetic field and collected them on a glass plate. Classically the atoms' magnets should point every way, giving a smear. Instead the beam split into two separate spots. A few years later this was explained by electron **spin** 自旋, which has only two values along any axis. Spin one-half is described by the **Pauli matrices** 泡利矩阵.

GRE Physics quantum questions ask you to:

- Find reflection and **transmission** 透射 at a potential step, using the **probability current** 概率流.
- Estimate **tunnelling** 隧穿 through a barrier.
- Count the **degeneracy** 简并度 of the 3D oscillator.
- Multiply Pauli matrices and find spin eigenvalues.

Steps and wave numbers

For a particle with $E > V$, the wavefunction is $e^{\pm ikx}$ with $k = \sqrt{2m(E-V)}/\hbar$. At a step, match ψ and ψ', with only a transmitted wave on the far side.

$$r = \frac{k_1 - k_2}{k_1 + k_2}, \qquad t = \frac{2k_1}{k_1 + k_2}$$

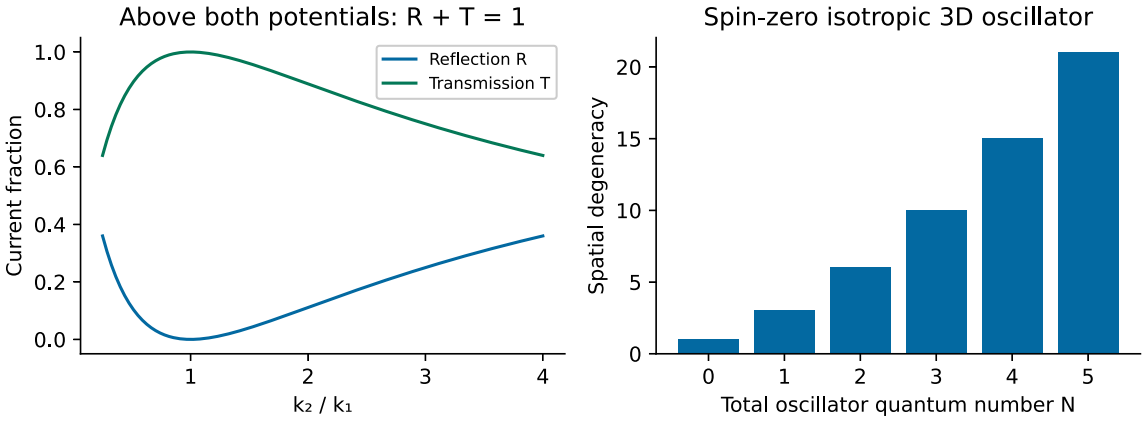
- A particle partly reflects even from a downward step, where nothing classical would turn it back.
- When $k_2 = k_1$ (no step), $r = 0$.

Probability current

The transmission probability carries a factor k_2/k_1, because the probability current is proportional to k:

$$R = |r|^2, \qquad T = \frac{k_2}{k_1}|t|^2, \qquad R + T = 1$$

- Without the factor, $R + |t|^2$ does not add to 1.



$R + T = 1$, with $R = 0$ when $k_2 = k_1$; level N has $(N + 1)(N + 2)/2$ states

Tunnelling through a barrier

If $E < V_0$ inside a barrier of width a, the wavefunction decays there as $e^{-\kappa x}$ with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$. For a wide barrier:

$$T \approx e^{-2\kappa a}$$

- Doubling the width squares the (small) transmission.
- Alpha decay and the scanning tunnelling microscope both depend on this.

The 3D isotropic oscillator

$E = \left(N + \frac{3}{2}\right)\hbar\omega$, with $N = n_x + n_y + n_z$.

- The number of ways to share N among three directions is $\dfrac{(N+1)(N+2)}{2}$: 1, 3, 6, 10, … (spin ignored).
- $N = 2$: the arrangements of $(2,0,0)$ and $(1,1,0)$, six states.

Spin and the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Each squares to the identity, so its eigenvalues are ± 1. Spin components are $S_i = \frac{\hbar}{2}\sigma_i$, giving $\pm\hbar/2$: Stern and Gerlach's two spots.
- $\sigma_x\sigma_y = i\sigma_z$, cyclic in $x \to y \to z$; reversing the order flips the sign.
- The eigenvectors of σ_x are $(1, \pm 1)/\sqrt{2}$.

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Worked example: a downward step

A particle meets a step where $k_2 = 3k_1$.

- $r = (1-3)/(1+3) = -\frac{1}{2}$ and $t = 2/(1+3) = \frac{1}{2}$.
- $R = \frac{1}{4}$ and $T = 3 \times \frac{1}{4} = \frac{3}{4}$, so $R + T = 1$.
- Using $|t|^2$ alone would give a total of $\frac{1}{2}$: probability would vanish.

Worked example: an electron tunnelling

A 1.0 eV electron meets a 5.0 eV barrier 0.50 nm wide.

- $\kappa = \sqrt{2(9.11 \times 10^{-31})(4.0 \times 1.6 \times 10^{-19})}/(1.05 \times 10^{-34}) = 1.0 \times 10^{10}$ m^{-1}.
- $2\kappa a = 2(1.0 \times 10^{10})(0.50 \times 10^{-9}) = 10$.
- $T \approx e^{-10} = 4.5 \times 10^{-5}$: about one electron in 22,000 gets through.

Common errors

- Forgetting the factor k_2/k_1 in the transmission.
- Saying a particle cannot reflect from a downward step.
- Counting the oscillator degeneracy as $N + 1$ (that is the 2D answer).
- Writing $\sigma_y \sigma_x = i\sigma_z$; it is $-i\sigma_z$.
- Thinking a spin component can take any value between $-\hbar/2$ and $+\hbar/2$.

Key terms

spin: The intrinsic angular momentum of a particle; for an electron, $\pm\hbar/2$ along any axis.

Pauli matrices: The three 2 by 2 matrices that represent spin one-half.

transmission: The fraction of particles that pass a step or barrier.

probability current: The flow of probability per unit time, proportional to k for a plane wave.

tunnelling: Passing through a region where the energy is below the potential.

degeneracy: The number of states that share one energy level.

Check yourself

- How many states does the $N = 3$ level of the 3D oscillator have (no spin)?
- What is $\sigma_y \sigma_x$?
- What is R when $k_2 = k_1$?

Answers: $(4)(5)/2 = 10$. $-i\sigma_z$. Zero.

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Nuclear binding, particle families and Hall carriers

GRE Subject Test · GRE Physics

In 1879 Edwin Hall, a graduate student at Johns Hopkins University, passed a current along a thin strip of gold leaf placed in a strong magnetic field. A sensitive galvanometer connected across the strip, at right angles to the current, showed a small steady voltage. The magnetic force was pushing the moving charges to one side. The sign of this **Hall effect** 霍尔效应 voltage later showed that in some materials the carriers behave as positive charges. Measuring carriers, nuclei and particles by what they are made of is the theme of this topic.

GRE Physics nuclear and particle questions ask you to:

- Compare stability by **binding energy per nucleon** 平均核子结合能.
- Find reaction energies (**Q value** Q 值).
- Sort particles into **leptons** 轻子, **mesons** 介子 and **baryons** 重子 by their **quarks** 夸克.
- Use the Hall coefficient to find the sign and density of carriers.

Binding energy per nucleon

Binding energy is the mass deficit times c^2. Stability follows the binding energy per nucleon B/A, not the total.

- B/A rises steeply for light nuclei, peaks near iron and nickel (about 8.8 MeV), and falls slowly for heavy ones.
- $B = 24$ MeV with $A = 4$ gives 6 MeV per nucleon; $B = 80$ MeV with $A = 10$ gives 8 MeV.
- Fusion of light nuclei and fission of heavy ones both move towards the peak and release energy.

Reaction energies

$$Q = (m_{\text{initial}} - m_{\text{final}})\, c^2$$

- Positive Q releases energy, shared as kinetic energy of the products.
- With atomic masses, the electron masses cancel as long as the charge balances.

Particle families

Classify a particle by its constituents.

- Leptons (electron, muon, tau and their neutrinos) contain no quarks. The muon is a heavy lepton, not a meson.

- Mesons are a quark and an antiquark: pions, kaons.
- Baryons are three quarks: proton uud, neutron udd.

Particle families in the ordinary constituent model

Examples	Ordinary constituent model	Family
Electron / muon	None	Lepton
Pion / kaon	Quark + antiquark	Meson
Proton / neutron	Three quarks	Baryon

Classify a particle by its constituents: leptons, mesons or baryons

Conservation checks

A reaction must conserve charge, energy, momentum, baryon number and each lepton number.

- Quark charges: $u = +\frac{2}{3}e$, $d = -\frac{1}{3}e$, so $uud = +e$ and $udd = 0$.
- $n \to p + e^- + \bar{\nu}_e$ balances charge, baryon number and electron-lepton number.
- A free proton cannot decay to a neutron: the neutron is heavier.

The Hall effect

A current I flows along a strip of thickness t in a field B perpendicular to the strip. Charges are pushed sideways until the Hall field balances the magnetic force.

$$V_H = \frac{IB}{nqt}, \qquad R_H = \frac{1}{nq}$$

- R_H is negative for electrons and positive for holes. Conductivity alone cannot tell them apart.
- Fewer carriers give a larger Hall voltage, so semiconductors show the effect far more strongly than metals.
- With both electrons and holes present, a model weighted by their mobilities is needed.

Worked example: deuterium–tritium fusion

$^{2}\mathrm{H} + {}^{3}\mathrm{H} \to {}^{4}\mathrm{He} + \mathrm{n}$, with atomic masses 2.014102 u, 3.016049 u, 4.002603 u and 1.008665 u.

- Initial mass: 5.030151 u. Final mass: 5.011268 u.
- $Q = 0.018883 \times 931.5 = 17.6$ MeV.
- Momentum sharing gives most of this (about 14.1 MeV) to the light neutron.

Worked example: the Hall voltage in copper

A copper strip 0.10 mm thick carries 5.0 A in a 1.0 T field; copper has $n = 8.5 \times 10^{28}$ m^{-3}.

- $V_H = \dfrac{IB}{net} = \dfrac{(5.0)(1.0)}{(8.5 \times 10^{28})(1.6 \times 10^{-19})(1.0 \times 10^{-4})} = 3.7 \times 10^{-6}$ V.
- Only a few microvolts, because copper has so many carriers.
- A semiconductor with 10^{22} carriers per m^3 would give a voltage about ten million times larger.

Common errors

- Judging stability by total binding energy instead of B/A.
- Calling the muon a meson.
- Forgetting the antineutrino in beta decay.
- Reading the sign of a Hall voltage without stating the current, field and lead directions.
- Expecting conductivity to reveal the sign of the carriers.

Key terms

Hall effect: A sideways voltage across a conductor carrying current in a magnetic field.

binding energy per nucleon: The binding energy divided by the number of nucleons; it measures stability.

Q value: The energy released by a reaction, from the mass difference.

lepton: A particle with no quarks, such as the electron, muon or a neutrino.

meson: A particle made of a quark and an antiquark.

baryon: A particle made of three quarks.

quark: An elementary particle that makes up protons, neutrons and other hadrons.

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Check yourself

- A nucleus with $A = 10$ has $B = 80$ MeV. What is its binding energy per nucleon?
- What is the charge of a uuu baryon?
- A Hall coefficient is positive. What are the carriers?

Answers: 8 MeV. $+2e$. Holes (positive carriers).

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Constraints, variational equations and cyclic coordinates

GRE Subject Test · GRE Physics

In 1788 Joseph-Louis Lagrange published *Mécanique analytique*, a complete treatment of mechanics. Its preface proudly warned that the reader would find no diagrams in it: only algebra. Instead of drawing every force, Lagrange wrote the kinetic and potential energy in coordinates that already obey the **constraints** 约束, and let one equation do the rest. Forces such as a string's tension, which only enforce a constraint, never appear.

GRE Physics questions on the Lagrangian method ask you to:

- Choose **generalised coordinates** 广义坐标 that respect the constraints.
- Write and differentiate the **Euler–Lagrange equation** 欧拉 – 拉格朗日方程 correctly.
- Spot a **cyclic coordinate** 循环坐标 and its conserved momentum.
- Find small-oscillation frequencies and equilibrium points.

Coordinates that respect constraints

A pendulum bob on a string of fixed length l needs one coordinate, the angle θ: $x = l\sin\theta$, $y = -l\cos\theta$.

- Then $v^2 = l^2\dot{\theta}^2$, $T = \frac{1}{2}ml^2\dot{\theta}^2$ and $V = mgl(1-\cos\theta)$.
- A **holonomic** 完整的 constraint is a relation among the coordinates; each one removes a coordinate.
- Constraints on velocities, such as rolling on a surface, may not reduce to coordinate relations and need more care.

The Euler–Lagrange equation

With $L = T - V$, each coordinate obeys:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Pendulum: $ml^2\ddot{\theta} + mgl\sin\theta = 0$, with no tension in sight.
- Differentiate every factor that changes with time: $\dfrac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$.

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Cyclic coordinates

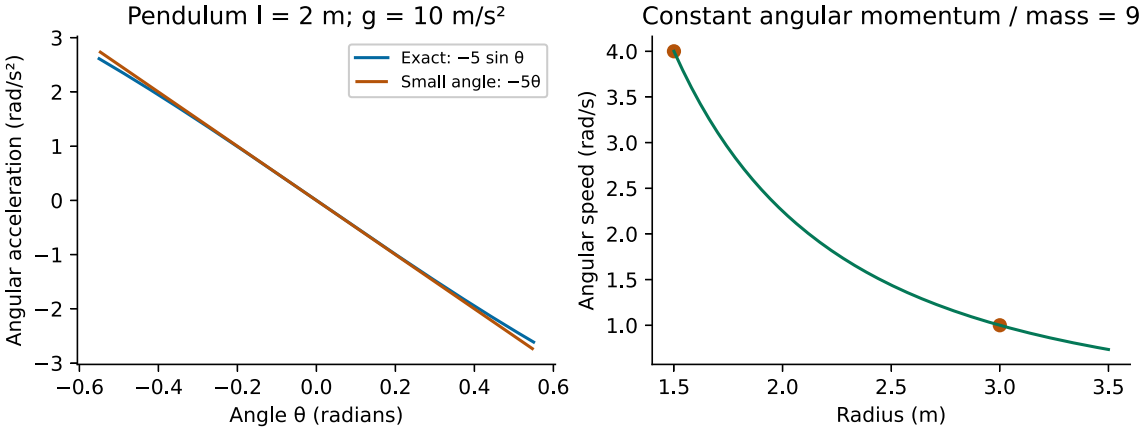
If a coordinate does not appear in L (only its rate does), its momentum $p = \partial L/\partial \dot{q}$ is conserved.

- Central force: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ contains no θ, so $p_\theta = mr^2\dot{\theta}$ is constant.
- Halving r multiplies $\dot{\theta}$ by 4.
- The radial equation is $m\ddot{r} = mr\dot{\theta}^2 - V'(r)$: the first term is the centrifugal effect of the rotation.

Small oscillations

Near a stable equilibrium q_0, the frequency is $\omega = \sqrt{V''(q_0)/M}$, where M is the coefficient of $\frac{1}{2}\dot{q}^2$ in T.

- Pendulum: $V'' = mgl$ and $M = ml^2$, so $\omega = \sqrt{g/l}$.
- If $V''(q_0) < 0$, the equilibrium is unstable: report the instability, not an imaginary frequency.



Small angles make the pendulum linear ($\ddot{\theta} \approx -5\theta$); constant angular momentum makes ω fall as $1/r^2$

Worked example: a 2 m pendulum

$l = 2.0$ m and $g = 10$ m/s^2, so $\ddot{\theta} = -5\sin\theta$.

- At $\theta = 0.10$ rad: $\ddot{\theta} = -0.499$ rad/s^2, against -0.500 from the small-angle form.
- Small-angle period: $2\pi/\sqrt{5} = 2.81$ s.
- At $\theta = 0.5$ rad the two forms differ by 4%, so large swings run slow.

Worked example: a bead on a spinning hoop

A bead slides without friction on a vertical hoop of radius R that spins at Ω about its vertical diameter. Measure θ from the bottom.

- $L = \tfrac{1}{2}mR^2(\dot\theta^2 + \Omega^2 \sin^2\theta) + mgR\cos\theta$.
- Euler–Lagrange: $R\ddot\theta = \sin\theta\,(\Omega^2 R\cos\theta - g)$. Equilibria: $\theta = 0$, or $\cos\theta = g/(\Omega^2 R)$ when $\Omega^2 > g/R$.
- With $R = 0.20$ m, $\Omega = 10$ rad/s and $g = 10$ m/s^2: $\cos\theta = 10/20 = 0.5$, so the bead rides at 60°.

Common errors

- Writing $L = T + V$.
- Treating r as constant when differentiating $mr^2\dot\theta$.
- Calling a coordinate cyclic when only its rate is missing from L (it must be the coordinate that is missing).
- Using the frequency formula at an unstable equilibrium.
- Adding constraint forces back into the Lagrangian.

Key terms

constraint: A condition that limits how a system can move.

generalised coordinates: Independent variables that fix a system's configuration.

Euler–Lagrange equation: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q} - \dfrac{\partial L}{\partial q} = 0$.

cyclic coordinate: A coordinate that does not appear in the Lagrangian; its momentum is conserved.

holonomic: Expressible as a relation among the coordinates (and time).

Check yourself

- For $L = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - V(r)$, which momentum is conserved?
- A particle in a central force moves from r to $r/2$. What happens to $\dot\theta$?
- For the spinning hoop with $\Omega^2 < g/R$, where can the bead rest stably?

Answers: $p_\theta = mr^2\dot\theta$. It becomes 4 times larger. Only at the bottom, $\theta = 0$.

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Accelerating and rotating reference frames

GRE Subject Test · GRE Physics

In 1851 Léon Foucault hung a 28 kg iron ball on a 67 m wire from the dome of the Panthéon in Paris and set it swinging. Over an hour, the line of its swing turned slowly clockwise, by about 11°, scraping a new line in a ring of sand. Nothing pushed the pendulum sideways. The floor, and the whole of Paris, was turning under it. Seen from the turning Earth, the motion needs extra terms: **apparent forces** 惯性力 such as the **Coriolis force** 科里奥利力.

GRE Physics questions on non-inertial frames ask you to:

- Handle a frame that accelerates in a straight line.
- Use the Coriolis, **centrifugal** 离心的 and Euler terms in a **rotating frame** 旋转参考系.
- Get the directions of these terms right.
- Check that the frame terms vanish when the frame stops accelerating.

Translating frames

In a frame accelerating at $\mathbf{A}$ relative to an **inertial frame** 惯性参考系:

$$m\mathbf{a}' = \mathbf{F}_{\text{real}} - m\mathbf{A}$$

- In a lift accelerating upward at a, the floor pushes with $N = m(g + a)$.
- This is the same physics as the ground-frame equation $N - mg = ma$, only written from inside the lift.

Rotating frames

In a frame rotating at $\boldsymbol{\Omega}$:

$$m\mathbf{a}' = \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}' - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - m\dot{\boldsymbol{\Omega}} \times \mathbf{r}$$

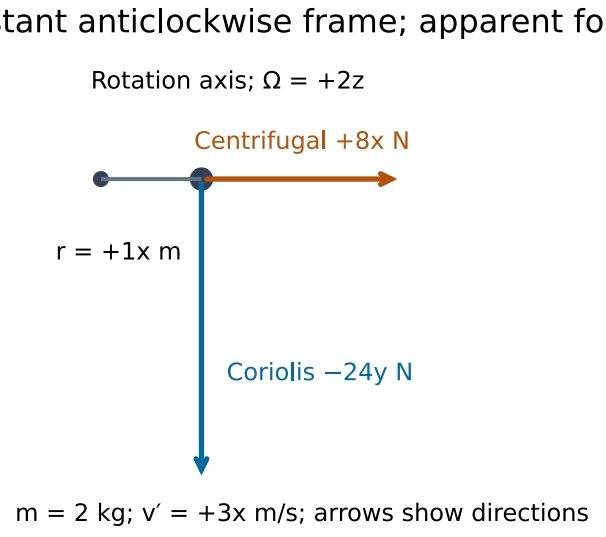
- Coriolis term $-2m\boldsymbol{\Omega} \times \mathbf{v}'$: needs a velocity $\mathbf{v}'$ relative to the rotating frame.
- Centrifugal term: points outward from the axis, with size $m\Omega^2 r_\perp$.
- **Euler force** 欧拉力 $-m\dot{\boldsymbol{\Omega}} \times \mathbf{r}$: only when the rotation rate changes.

Getting directions right

Take $\boldsymbol{\Omega}$ along $+z$ (anticlockwise seen from above) and a particle on the $+x$ axis moving outward along $+x$.

- Centrifugal: along $+x$, away from the axis.

- $\boldsymbol{\Omega} \times \mathbf{v}'$ points along $+y$, so the Coriolis force points along $-y$: to the right of the motion.
- On the northern half of the Earth, moving bodies are deflected to the right; on the southern half, to the left.



Centrifugal force $m\Omega^2 r = 8$ N; Coriolis force $-2m\boldsymbol{\Omega} \times \mathbf{v}' = -24$ N on y

The rotating Earth

The Earth turns once a day: $\Omega = 7.3 \times 10^{-5}$ rad/s.

- At latitude λ, a Foucault pendulum's plane turns once in $24\ \text{h}/\sin\lambda$: once a day at a pole, never at the equator.
- The centrifugal term makes the effective g at the equator about $0.034\ \text{m/s}^2$ smaller than it would otherwise be.

Checks

- A body at rest on a steady platform has no Coriolis force; the real force on it is $m\Omega^2 r$ inward.
- The Coriolis force is perpendicular to $\mathbf{v}'$, so it does no work.
- Set $\Omega \to 0$ and $\mathbf{A} \to 0$: ordinary Newtonian motion must come back.

Worked example: a lift and a platform

- A 10 kg passenger in a lift accelerating upward at 2.0 m/s^2 ($g = 10$): $N = 10(10 + 2.0) = 120$ N.
- A 2.0 kg particle 1.0 m from the axis of a platform turning at 2.0 rad/s, moving outward at 3.0 m/s: centrifugal $2.0 \times 2.0^2 \times 1.0 = 8.0$ N along $+x$.
- Coriolis: $2 \times 2.0 \times 2.0 \times 3.0 = 24$ N along $-y$. With no real force, $\mathbf{a}' = (4.0, -12)$ m/s^2.

Worked example: Foucault's pendulum in Paris

Paris is at latitude 48.9°.

- Period of the plane's rotation: $24\ \text{h} / \sin 48.9° = 24/0.753 = 31.9$ h.
- Rate: $360°/31.9\ \text{h} = 11.3°$ per hour, clockwise seen from above.
- At the North Pole the plane would turn once in a day (strictly, a sidereal day of 23.9 h).

Common errors

- Including a Coriolis force on a body at rest in the rotating frame.
- Using the velocity relative to the ground instead of relative to the rotating frame.
- Getting the Coriolis direction backwards; work out $\boldsymbol{\Omega} \times \mathbf{v}'$ and then change its sign.
- Adding centrifugal force in an inertial frame.
- Forgetting the $\sin\lambda$ factor for a Foucault pendulum.

Key terms

apparent force: A term $-m\mathbf{A}$ or a rotation term that appears only in a non-inertial frame.

Coriolis force: The rotating-frame term $-2m\boldsymbol{\Omega} \times \mathbf{v}'$ on a moving body.

centrifugal: Directed outward, away from the axis of rotation.

rotating frame: A reference frame that turns relative to an inertial frame.

inertial frame: A frame in which a body with no net force moves in a straight line at constant speed.

Euler force: The rotating-frame term $-m\dot{\boldsymbol{\Omega}} \times \mathbf{r}$, present only when the rotation rate changes.

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Check yourself

- A 60 kg person stands in a lift accelerating downward at 2.0 m/s^2 ($g = 10$). What does the floor push with?
- How long does a Foucault pendulum take to turn once at the equator?
- What is the Coriolis force on a particle at rest in a rotating frame?

Answers: $60(10 - 2.0) = 480$ N. It never turns there. Zero.

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Fluid continuity, pressure energy and viscous flow

GRE Subject Test · GRE Physics

In 1883 Osborne Reynolds, in Manchester, ran water from a large tank through a long glass tube and fed a thin streak of dye into the entrance. At low speeds the dye stayed a straight, sharp line: the flow was **laminar** 层流. As he opened the valve further, the streak suddenly broke up and filled the tube with swirls: the flow had become **turbulent** 湍流. The change depended on one combination of speed, tube size, density and **viscosity** 黏度, now called the **Reynolds number** 雷诺数.

GRE Physics fluids questions ask you to:

- Use **continuity** 连续性 and **hydrostatic pressure** 静水压强.
- Apply Bernoulli's equation along a **streamline** 流线, and know its conditions.
- Use Poiseuille's law for viscous flow.
- Judge laminar or turbulent flow from the Reynolds number.

Continuity and hydrostatics

For steady, incompressible flow, the volume flow rate $Q = Av$ is the same everywhere along a pipe.

- Halving the radius quarters the area and multiplies the speed by 4.
- In a fluid at rest, the pressure at depth h is $p = p_0 + \rho g h$.
- Use absolute or gauge pressure consistently. Every 10 m of water adds about one atmosphere.

Bernoulli's equation

Along a streamline of steady, incompressible flow without viscosity or pumps:

$$p + \tfrac{1}{2}\rho v^2 + \rho g z = \text{constant}$$

- Each term is an energy per unit volume.
- Faster flow at the same height means lower pressure (the Venturi effect).
- Water leaving a hole at depth h below an open surface has $v = \sqrt{2gh}$ (Torricelli's law).

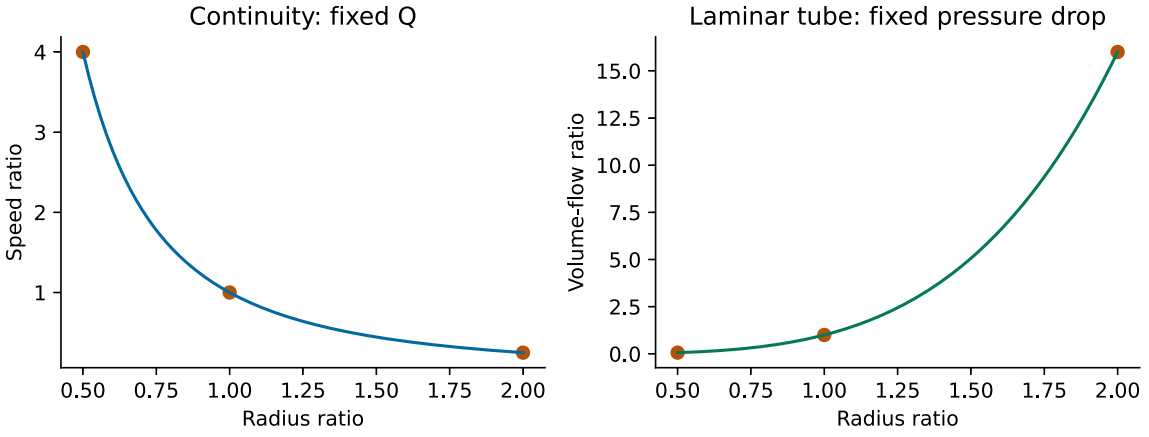
Viscous flow: Poiseuille's law

For laminar flow through a tube of radius R and length L, driven by a pressure difference Δp:

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$$Q = \frac{\pi R^4 \, \Delta p}{8 \eta L}$$

- Doubling R at the same Δp multiplies the flow by 16.
- This compares different tubes at the same pressure drop; continuity compares parts of one pipe at the same Q.



Continuity gives $v \propto 1/r^2$; Poiseuille flow gives $Q \propto r^4$

The Reynolds number

$$\mathrm{Re} = \frac{\rho v D}{\eta}$$

- It compares inertia with viscosity and has no units.
- Low Re: viscous, laminar flow. High Re: inertia dominates and turbulence becomes possible. For a round pipe the change is near $\mathrm{Re} \approx 2000$, but the threshold depends on the geometry.
- Poiseuille's R^4 law fails once the flow is turbulent.

Worked example: a narrowing pipe

Water ($\rho = 1000$ kg/m^3) flows at 1.0 m/s through a 4.0 cm^2 pipe into a 1.0 cm^2 section at the same height.

- Continuity: $v_2 = 4 \times 1.0 = 4.0$ m/s and $Q = 4.0 \times 10^{-4}$ m^3/s.
- Bernoulli: $p_1 - p_2 = \frac{1}{2}(1000)(4.0^2 - 1.0^2) = 7500$ Pa.
- With viscosity, the pressure would also fall along each straight section.

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Worked example: a narrowed artery and a water pipe

- An artery's radius shrinks by 20% at the same pressure difference: Q falls to $0.8^4 = 0.41$ of its old value, a drop of 59%.
- Water ($\eta = 1.0 \times 10^{-3}$ Pa s) at 0.50 m/s in a 2.0 cm pipe: $\mathrm{Re} = (1000)(0.50)(0.020)/(1.0 \times 10^{-3}) = 10{,}000$.
- That is well above 2000, so the flow is probably turbulent and Poiseuille's law does not apply.

Common errors

- Using Bernoulli's equation across a pump or with strong viscous losses.
- Confusing continuity (Q fixed) with Poiseuille (Δp fixed).
- Mixing gauge and absolute pressure.
- Using the radius instead of the diameter in the Reynolds number.
- Applying the R^4 law to turbulent flow.

Key terms

laminar: Flowing in smooth layers that do not mix.

turbulent: Flowing with irregular swirls and mixing.

viscosity: A fluid's internal friction, its resistance to flowing.

Reynolds number: The dimensionless ratio of inertial to viscous effects in a flow.

continuity: For steady incompressible flow, Av is the same along a pipe.

hydrostatic pressure: The pressure in a fluid at rest, which grows with depth as $\rho g h$.

streamline: The path followed by a small piece of fluid in steady flow.

Check yourself

- Water leaves a hole 5.0 m below the surface of an open tank. How fast?
- What is the gauge pressure 10 m below the surface of water?
- Within one pipe, the radius doubles. What happens to the speed?

Answers: $\sqrt{2(9.8)(5.0)} = 9.9$ m/s. $1000 \times 9.8 \times 10 = 98$ kPa, about one atmosphere. It falls to a quarter.

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Polarisation, magnetisation and material fields

GRE Subject Test · GRE Physics

In 1837 Michael Faraday built two capacitors from pairs of concentric brass spheres. He filled the gap of one with shellac, sulphur or glass, left the other with air, and compared how much charge each held at the same voltage. The filled one always held more. He called the ratio the "specific inductive capacity"; today it is the relative **permittivity** 介电常数 of a **dielectric** 电介质. The molecules of the insulator line up with the field, and their **polarisation** 极化强度 partly cancels it.

GRE Physics questions on fields in matter ask you to:

- Use **P**, **D** and **bound charge** 束缚电荷.
- Decide whether charge or voltage is fixed when a dielectric is inserted.
- Apply the field conditions at an interface.
- Use **magnetisation** 磁化强度, **H**, **B** and the **susceptibility** 磁化率.

Polarisation and the D field

P is the dipole moment per unit volume ($\mathrm{C/m^2}$). The displacement field collects the free charge only:

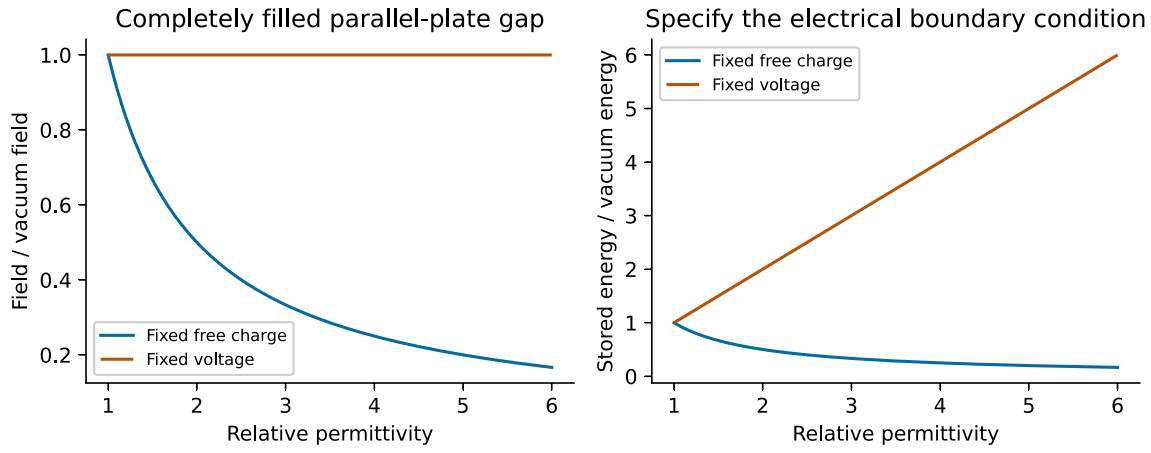
$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \nabla \cdot \mathbf{D} = \rho_{\text{free}}$$

- Bound charge: $\rho_b = -\nabla \cdot \mathbf{P}$ inside and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ on surfaces.
- In a linear dielectric, $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$. The field inside is reduced by ε_r, not removed.

Fixed charge or fixed voltage

Filling a capacitor with a dielectric multiplies C by ε_r. What happens next depends on the battery.

- Disconnected (fixed Q): V, E and the energy $U = Q^2/2C$ all fall by ε_r.
- Connected (fixed V): Q and the energy $U = \frac{1}{2}CV^2$ rise by ε_r; $E = V/d$ is unchanged.
- The same slab gives opposite energy changes in the two cases.



At fixed charge a dielectric lowers E and energy by κ; at fixed voltage the energy rises by κ

Interfaces

At a boundary between two dielectrics:

- The normal component of **D** jumps by the free surface charge (zero if there is none).
- The tangential component of **E** is continuous.
- So when ε changes, the normal component of **E** changes, and field lines bend.

Magnetisation, H and B

M is the magnetic moment per unit volume (A/m).

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \qquad \mathbf{M} = \chi_m \mathbf{H} \text{ (linear materials)}$$

- In a long solenoid, $H = nI$ whatever the core; the core changes B through M.
- Diamagnets have a small negative χ_m; paramagnets a small positive one.
- Ferromagnets have huge, non-linear magnetisation, saturate, and show **hysteresis** 磁滞: their magnetisation depends on their history.

Worked example: a filled capacitor

A capacitor with free charge density σ is filled with a dielectric of $\varepsilon_r = 4$, with the battery disconnected.

- $D = \sigma$, so $E = \sigma/(4\varepsilon_0)$, a quarter of the empty value.
- $P = \varepsilon_0(\varepsilon_r - 1)E = \frac{3}{4}\sigma$; the bound charge $-\frac{3}{4}\sigma$ sits next to the positive plate.
- The voltage and stored energy both fall to a quarter.

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Worked example: battery on or off

An empty 10 pF capacitor is charged to 100 V, so $Q = 1.0$ nC and $U = 50$ nJ. A slab with $\varepsilon_r = 4$ then fills it.

- Disconnected: $C = 40$ pF, $V = 25$ V, $U = 12.5$ nJ. The slab is pulled in, and the field does work on it.
- Connected: $Q = 4.0$ nC and $U = 200$ nJ. The battery supplies $QV = 300$ nJ of extra energy: 150 nJ is stored, and 150 nJ is the work done pulling the slab in.
- A solenoid with $H = 400$ A/m and a core of $\chi_m = 0.5$: $M = 200$ A/m and $B = \mu_0(600) = 0.75$ mT.

Common errors

- Saying the field inside a dielectric is zero; that is a conductor.
- Assuming the energy always falls when a dielectric is inserted.
- Using total charge in $\nabla \cdot \mathbf{D}$; it counts free charge only.
- Taking tangential **D** or normal **E** as continuous at an interface.
- Treating a ferromagnet as linear.

Key terms

permittivity: How strongly a material reduces an electric field; $\varepsilon = \varepsilon_r \varepsilon_0$.

dielectric: An insulator that becomes polarised in an electric field.

polarisation: Electric dipole moment per unit volume.

bound charge: Charge that appears on or in a dielectric because of its polarisation.

magnetisation: Magnetic dipole moment per unit volume.

susceptibility: The ratio $\chi_m = M/H$ for a linear magnetic material.

hysteresis: Dependence of a material's state on its past, as in the magnetisation of a ferromagnet.

Check yourself

- A disconnected capacitor is filled with a slab of $\varepsilon_r = 3$. What happens to its voltage?
- A diamagnet has $\chi_m = -1.0 \times 10^{-5}$ in $H = 1000$ A/m. What is M?
- Is the normal component of **D** continuous at an interface with no free charge?

Answers: it falls to a third. -0.010 A/m. Yes.

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RC and RL transients, impedance and resonance

GRE Subject Test · GRE Physics

In 1887 Heinrich Hertz, in Karlsruhe, made sparks jump across a small gap in a circuit of wire and metal plates: in effect an inductor and a capacitor, oscillating about a hundred million times a second. Across the room, a simple loop of wire with its own tiny gap showed faint sparks too, strongest when its size matched the first circuit. That matching is **resonance** 共振, and Hertz had sent and received the first radio waves. Every circuit question about capacitors and inductors comes back to time constants and resonance.

GRE Physics circuit questions ask you to:

- Solve RC and RL **transients** 暂态 with their time constants.
- Know what cannot change suddenly, and what happens long after a switch.
- Use complex **impedance** 阻抗, **reactance** 电抗 and **phase** 相位.
- Find the resonant frequency, current and **quality factor** 品质因数.

RC transients

A capacitor charging through R from a source V_s obeys $R\,dq/dt + q/C = V_s$, with $\tau = RC$.

- From zero charge: $V_C = V_s(1 - e^{-t/\tau})$ and $I = (V_s/R)e^{-t/\tau}$.
- After one τ, the capacitor has reached 63% of its final voltage; after 5τ, more than 99%.

RL transients

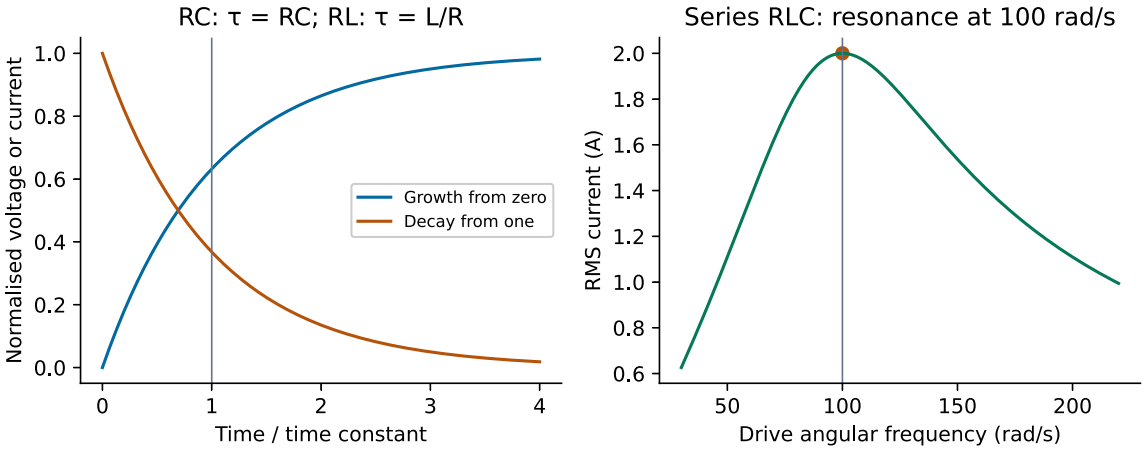
An inductor in series with R obeys $L\,dI/dt + RI = V_s$, with $\tau = L/R$.

- From zero current: $I = (V_s/R)(1 - e^{-t/\tau})$.
- $L = 0.40$ H and $R = 2.0\ \Omega$ give $\tau = 0.20$ s.

What cannot jump

- A capacitor's voltage and an inductor's current cannot change instantly.
- Just after a switch closes, an uncharged capacitor acts like a wire and an inductor with no current acts like a break.
- Long afterwards, the capacitor blocks steady current (open circuit) and the inductor passes it freely (wire). The stored energies $\frac{1}{2}CV^2$ and $\frac{1}{2}LI^2$ end as heat in the resistors.

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$\tau = RC$ or L/R; a series RLC circuit draws its largest current at resonance

Impedance and phase

For AC at angular frequency ω: $Z_R = R$, $Z_L = i\omega L$ and $Z_C = 1/(i\omega C)$. In series:

$$Z = R + iX, \qquad X = \omega L - \frac{1}{\omega C}, \qquad |I| = \frac{|V|}{|Z|}, \qquad \tan\varphi = \frac{X}{R}$$

- The current lags the voltage when $X > 0$ (inductive) and leads it when $X < 0$ (capacitive).
- Never add R, ωL and $1/\omega C$ as plain numbers; they are at right angles.

Resonance

At $\omega_0 = 1/\sqrt{LC}$ the reactances cancel: $X = 0$, $|Z| = R$, and the current is largest and in phase with the voltage.

- Only the resistor takes power: $P = I_{\text{rms}}^2 R$.
- The quality factor $Q = \omega_0 L/R$ measures sharpness: the resonance peak is about ω_0/Q wide.
- The free, damped oscillation frequency $\sqrt{1/LC - (R/2L)^2}$ is a different quantity.

Worked example: charging a capacitor

$R = 2000\ \Omega$, $C = 1.0$ mF and $V_s = 12$ V, starting uncharged.

- $\tau = RC = 2.0$ s, and the starting current is $12/2000 = 6.0$ mA.
- At $t = \tau$: $V_C = 12(1 - e^{-1}) = 7.6$ V and $I = 6.0e^{-1} = 2.2$ mA.
- Energy finally stored: $\frac{1}{2}(1.0 \times 10^{-3})(12)^2 = 72$ mJ; the resistor turns another 72 mJ into heat while charging.

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Worked example: a series RLC circuit

$R = 10\ \Omega$, $L = 0.10$ H, $C = 1.0$ mF, driven at $V_{\text{rms}} = 20$ V.

- $\omega_0 = 1/\sqrt{(0.10)(1.0 \times 10^{-3})} = 100$ rad/s. There $I_{\text{rms}} = 20/10 = 2.0$ A and $P = 40$ W. $Q = (100)(0.10)/10 = 1$.
- At $\omega = 200$ rad/s: $X = 20 - 5 = 15\ \Omega$, $|Z| = \sqrt{10^2 + 15^2} = 18\ \Omega$, so $I_{\text{rms}} = 1.1$ A.
- There $X > 0$, so the current lags by $\tan^{-1}(1.5) = 56°$.

Common errors

- Letting a capacitor's voltage or an inductor's current jump at a switch.
- Using $\tau = RL$ or $\tau = R/L$; it is L/R.
- Adding reactances and resistance as plain numbers.
- Getting the sign of the phase wrong: inductive means current lags.
- Confusing the driven resonance with the free damped frequency.

Key terms

resonance: The driving frequency at which a circuit's reactances cancel and the current is largest.

transient: The short-lived change in a circuit after a switch, before it settles.

impedance: The AC opposition to current, $Z = R + iX$.

reactance: The part of the impedance from inductors and capacitors, $\omega L - 1/\omega C$.

phase: The angle by which the current leads or lags the voltage.

quality factor: $Q = \omega_0 L/R$, a measure of how sharp a resonance is.

Check yourself

- A 10 V source drives an RL circuit with $L = 0.40$ H and $R = 2.0\ \Omega$. What is the current 0.20 s after switching on?
- What is the phase between current and voltage at resonance?
- Just after a switch closes, how does an uncharged capacitor behave?

Answers: $5.0(1 - e^{-1}) = 3.2$ A. Zero. Like a wire (a short circuit).

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Polarisation, analyser chains and phase

GRE Subject Test · GRE Physics

One evening in 1808, Étienne-Louis Malus looked through a crystal of calcite at the sunset reflected in the windows of the Luxembourg Palace in Paris. Calcite usually shows two images. As he turned the crystal, first one image and then the other grew faint. Reflected light, he realised, was partly polarised: its electric field preferred one direction. Polarisers, sunglasses and LCD screens all use this today.

GRE Physics polarisation questions ask you to:

- Apply **Malus's law** 马吕斯定律 through a chain of **polarisers** 偏振片.
- Treat **unpolarised** 非偏振的 light correctly at the first filter.
- Recognise linear, **circular polarisation** 圆偏振 and elliptical states, and what a **quarter-wave plate** 四分之一波片 does.
- Use **Brewster's angle** 布儒斯特角.

Malus's law, step by step

An ideal polariser passes the component of **E** along its axis, $E\cos\theta$. So the intensity becomes:

$$I = I_{\text{in}}\cos^2\theta$$

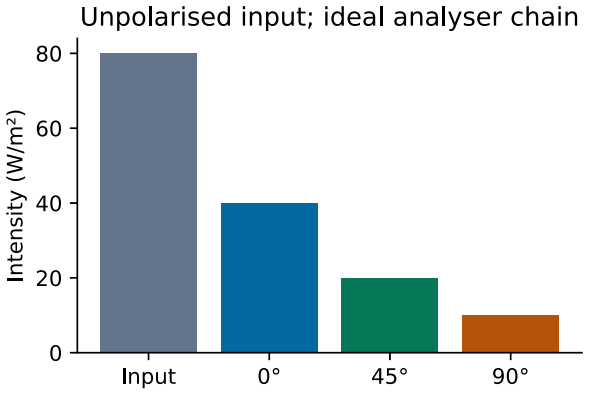
- θ is measured from the light's current polarisation, which is the axis of the previous polariser.
- Unpolarised light loses half its intensity at the first polariser, and leaves polarised along that axis. Only that first step uses the factor $\frac{1}{2}$.

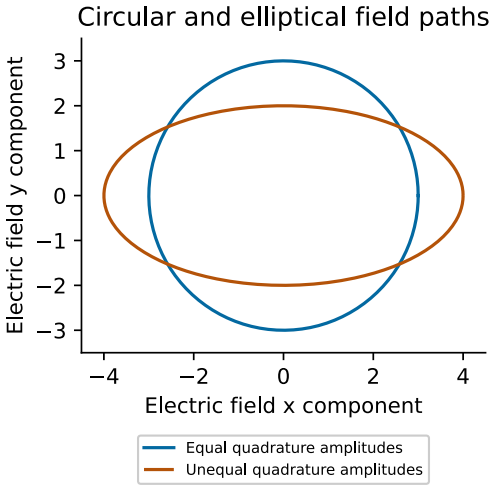
A chain of polarisers

Crossed polarisers (axes 90° apart) block everything. Put a third one between them at 45°, and light gets through.

- $I_0 \to I_0/2 \to (I_0/2)\cos^2 45^\circ = I_0/4 \to (I_0/4)\cos^2 45^\circ = I_0/8$.
- The middle filter does not add light; it turns the polarisation in two steps instead of one.

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The first polariser halves unpolarised light; each later one applies Malus's law, $\cos^2\theta$

Polarisation states

Write $E_x = A\cos\omega t$ and $E_y = B\cos(\omega t + \delta)$.

- $\delta = 0$ or π: linear polarisation.
- $A = B$ and $\delta = \pm\pi/2$: circular polarisation; the field vector traces a circle.
- Any other case: elliptical. Circular light through any ideal linear polariser keeps half its intensity.

Wave plates

A wave plate is made of a crystal whose refractive index differs along two axes, so it delays one component of the field relative to the other.

- A quarter-wave plate adds a phase of $\pi/2$. Linear light at 45° to its axes becomes circular; light along one axis stays linear.
- A polariser removes a component; a wave plate only shifts its phase.

Brewster's angle

At $\tan\theta_B = n_2/n_1$, the reflected light has no component in the plane of incidence (p), so it is fully s-polarised.

- At this angle the reflected and refracted rays are at 90° to each other.
- Air to glass ($n = 1.5$): $\theta_B = 56.3°$. Polarising sunglasses block the horizontal glare reflected from roads and water.
- Brewster's angle is not the critical angle, which needs light going from a denser medium to a less dense one.

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Worked example: three filters

80 W/m^2 of unpolarised light passes polarisers at 0°, 45° and 90°.

- After 0°: 40 W/m^2. After 45°: $40\cos^2 45° = 20$ W/m^2. After 90°: $20\cos^2 45° = 10$ W/m^2.
- Remove the middle filter and nothing gets through.
- Use six filters in 15° steps instead: each passes $\cos^2 15° = 0.933$, so $40 \times 0.933^6 = 26$ W/m^2 survive. More, smaller steps lose less.

Worked example: glare from water

Sunlight reflects from a lake ($n = 1.33$).

- $\theta_B = \tan^{-1}(1.33) = 53.1°$ from the vertical.
- Light reflected at that angle is polarised horizontally (s).
- A polariser with a vertical axis blocks it, which is why polarising sunglasses have vertical axes.

Common errors

- Halving the intensity at every polariser, not just the first.
- Measuring θ from the first polariser instead of the previous one.
- Thinking a wave plate absorbs light.
- Confusing Brewster's angle with the critical angle.
- Saying circular light is blocked by a linear polariser; half gets through.

Key terms

Malus's law: An ideal polariser passes $I\cos^2\theta$ of linearly polarised light.
polariser: A filter that passes the electric field along one axis.
unpolarised: With the field direction changing randomly, so no direction is preferred.
circular polarisation: Light whose field vector turns in a circle as the wave passes.
quarter-wave plate: A plate that delays one field component by a quarter of a cycle.
Brewster's angle: The angle of incidence at which reflected light is fully s-polarised.

Check yourself

- 100 W/m^2 of unpolarised light passes two polarisers 60° apart. What comes out?
- What is Brewster's angle from air into water ($n = 1.33$)?
- Linear light at 45° to the axes enters a quarter-wave plate. What comes out?

Answers: $50\cos^2 60° = 12.5$ W/m^2. 53.1°. Circularly polarised light.

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Diffraction envelopes and missing interference orders

GRE Subject Test · GRE Physics

In 1818 Augustin Fresnel entered a prize competition in Paris with a wave theory of diffraction. One judge, Siméon Poisson, thought he had found an absurd result in it: the theory predicted a bright spot at the very centre of the shadow of a small round disc. Another judge, François Arago, did the experiment, and the bright spot was there. Fresnel won. Diffraction patterns follow wave theory in every detail, including where bright fringes go missing.

GRE Physics diffraction questions ask you to:

- Use the single-slit intensity, its **envelope** 包络线 and its zeros.
- Combine slit width and slit spacing in the two-slit pattern.
- Predict **missing orders** 缺级.
- Use the **Rayleigh criterion** 瑞利判据 for a circular **aperture** 孔径.

The single-slit envelope

A slit of width a gives, in the **far field** 远场:

$$\frac{I}{I_0} = \left(\frac{\sin\beta}{\beta}\right)^2, \qquad \beta = \frac{\pi a \sin\theta}{\lambda}$$

- At $\theta = 0$ the limit is 1: the bright centre.
- Zeros at $a\sin\theta = m\lambda$, $m = \pm1, \pm2, \ldots$ At $\beta = \pi/2$, the intensity is $4/\pi^2 = 0.405$.
- The far field needs $L \gg a^2/\lambda$, or a lens that focuses the pattern.

Widths on a screen

- For small angles, the first zeros are at $y = \pm L\lambda/a$, so the central peak is $2L\lambda/a$ wide.
- For large angles, use $\theta = \sin^{-1}(m\lambda/a)$ and $y = L\tan\theta$.
- A narrower slit gives a wider pattern.

Two slits: envelope times fringes

Two slits of width a, spaced d apart, give the single-slit envelope multiplied by the two-slit fringes:

$$\frac{I}{I_0} = \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha, \qquad \alpha = \frac{\pi d \sin\theta}{\lambda}$$

- Fringes at $d\sin\theta = n\lambda$, spaced $\lambda L/d$ on a screen.

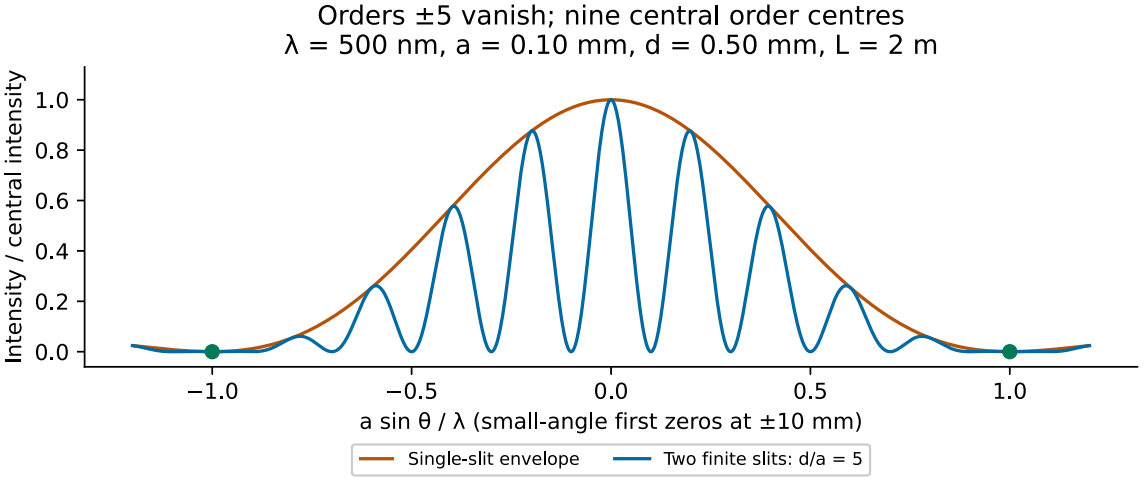
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- A larger d makes the fringes closer; a larger a makes the envelope narrower.

Missing orders

A fringe is missing when it falls exactly on an envelope zero: $n = m\,d/a$.

- If $d/a = r$ is a whole number, orders $\pm r, \pm 2r, \ldots$ vanish.
- The central envelope then holds $2r - 1$ fringes.



With $d/a = 5$ the fifth orders fall on envelope zeros, leaving nine central fringes

Circular apertures and resolution

A circular aperture of diameter D gives a central disc whose first dark ring is at:

$$\sin\theta \approx 1.22\,\frac{\lambda}{D}$$

- Rayleigh criterion: two point sources are just resolved when the centre of one pattern falls on the first dark ring of the other.
- Bigger telescopes resolve finer detail; shorter wavelengths do too.

Worked example: two slits with missing orders

$\lambda = 500$ nm, $a = 0.10$ mm, $d = 0.50$ mm, screen at $L = 2.0$ m.

- Envelope zeros: $y = \pm L\lambda/a = \pm 10$ mm. Fringe spacing: $\lambda L/d = 2.0$ mm.
- $d/a = 5$, so the fifth orders sit at ± 10 mm, exactly on the envelope zeros: missing.
- Nine fringes, $n = -4$ to $+4$, fill the central envelope.

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Worked example: seeing two headlights

The pupil of the eye is about 3.0 mm across; take $\lambda = 550$ nm.

- $\theta = 1.22 \times (5.5 \times 10^{-7})/(3.0 \times 10^{-3}) = 2.2 \times 10^{-4}$ rad.
- Headlights 1.5 m apart are resolved out to $1.5/(2.2 \times 10^{-4}) = 6.7$ km.
- In practice the eye does a little worse, because of imperfections and the spacing of its receptors.

Common errors

- Counting $m = 0$ as a single-slit zero.
- Mixing up the roles of a (envelope) and d (fringes).
- Expecting a fringe where the envelope is zero.
- Forgetting the factor 1.22 for a circular aperture.
- Using small-angle formulas at large angles.

Key terms

envelope: The single-slit pattern that sets the brightness of the two-slit fringes.

missing orders: Interference maxima that vanish because they fall on single-slit zeros.

far field: The region far enough from an aperture that the pattern keeps its shape as it spreads.

Rayleigh criterion: Two sources are just resolved when one's central maximum lies on the other's first minimum.

aperture: An opening that light passes through.

Check yourself

- What is the single-slit intensity at $\beta = \pi/2$, as a fraction of the centre?
- Two slits have $d/a = 3$. Which orders are missing, and how many fringes are in the central envelope?
- A slit is made narrower. What happens to the envelope?

Answers: $4/\pi^2 = 0.405$. $\pm 3, \pm 6, \ldots$; five fringes. It becomes wider.

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Signed lens and mirror images

GRE Subject Test · GRE Physics

In 1668 Isaac Newton built a small telescope only about 15 cm long that magnified about 40 times. Lens telescopes of the time blurred colours, because a lens bends different colours by different amounts. Newton used a curved mirror instead, which reflects all colours the same way, and a small flat mirror to send the light out to the side. Lenses and mirrors follow the same equation, provided you keep track of signs.

GRE Physics geometric-optics questions ask you to:

- Use the thin-lens and mirror equations with one sign convention.
- Tell a **real image** 实像 from a **virtual image** 虚像 from the signs.
- Find the **magnification** 放大率 and whether the image is upright.
- Follow light through two lenses.

The sign convention

Use the real-is-positive convention:

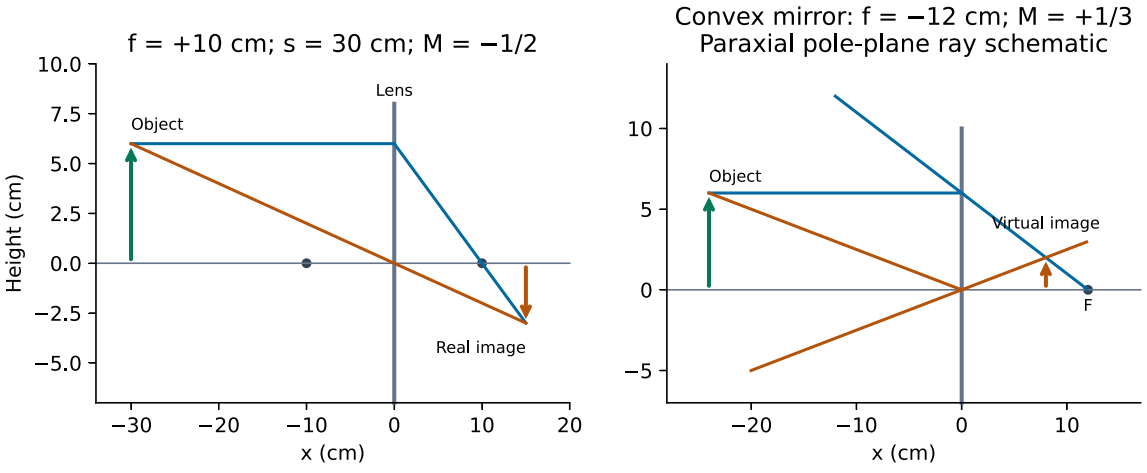
$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}, \qquad M = -\frac{s'}{s}$$

- A real object has $s > 0$; a real image has $s' > 0$; a virtual image has $s' < 0$.
- A **converging lens** 会聚透镜 has a positive **focal length** 焦距, $f > 0$; a **diverging lens** 发散透镜 has $f < 0$.
- A concave mirror has $f = R/2 > 0$; a convex mirror has $f < 0$.

Lens images

- Object beyond f: real, inverted image on the far side.
- Object inside f: virtual, upright, enlarged image on the same side (a magnifying glass).
- Object at f: parallel rays, no image. A diverging lens with a real object always gives a virtual, upright, smaller image.
- Ray rules: a ray parallel to the axis passes through the far focus; a ray through the centre goes straight on.

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Thin-lens equation: $s = 30$ cm, $f = 10$ cm gives $s' = 15$ cm and $M = -1/2$

Mirrors

Mirrors use the same equation and signs.

- A real mirror image forms in front of the mirror, where the light is; a virtual image is behind it.
- A convex mirror (car wing mirrors, shop mirrors) always gives a virtual, upright, smaller image, which shows a wide view.

Two lenses

Find the image of the first lens. Then use it as the object of the second lens, measuring from the second lens.

- If the first image lies beyond the second lens, the light is still converging when it arrives: a virtual object with $s_2 < 0$.
- The total magnification is $M = M_1 M_2$.
- Lens power in dioptres is $P = 1/f$ with f in metres; thin lenses in contact add powers.

Worked example: one lens, two positions

A lens has $f = 10$ cm.

- Object at 30 cm: $1/s' = 1/10 - 1/30$, so $s' = 15$ cm and $M = -\frac{1}{2}$: real, inverted, half size.
- Object at 5.0 cm: $1/s' = 1/10 - 1/5$, so $s' = -10$ cm and $M = +2$: virtual, upright, twice the size.
- A negative s' is a location (the image is on the incoming side), not a mistake to remove with an absolute value.

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Worked example: a mirror and a two-lens system

- Convex mirror, $f = -12$ cm, object at 24 cm: $1/s' = -1/12 - 1/24$, so $s' = -8.0$ cm and $M = +\frac{1}{3}$, upright, behind the mirror.
- Two lenses of $f = 10$ cm, at $x = 0$ and $x = 40$ cm, object at $x = -30$ cm. First image at $x = 15$ cm with $M_1 = -\frac{1}{2}$.
- Second lens: $s_2 = 40 - 15 = 25$ cm, $s_2' = 50/3 = 16.7$ cm, $M_2 = -\frac{2}{3}$. Total $M = +\frac{1}{3}$: upright and a third of the size.

Common errors

- Dropping the sign of s' and losing the virtual image.
- Using the lens spacing as the second object distance; measure from the first image.
- Forgetting that a convex mirror has a negative focal length.
- Adding magnifications instead of multiplying them.
- Putting a real mirror image behind the mirror.

Key terms

real image: An image formed where light actually meets; it can be shown on a screen.

virtual image: An image that light only seems to come from; it cannot be shown on a screen.

magnification: The ratio of image height to object height, $-s'/s$.

converging lens: A lens that brings parallel rays together; $f > 0$.

focal length: The distance from a lens or mirror to the point where parallel rays meet.

diverging lens: A lens that spreads parallel rays apart; $f < 0$.

Check yourself

- An object is 20 cm from a diverging lens with $f = -20$ cm. Where is the image, and what is M?
- A concave mirror has $R = 40$ cm and an object at 30 cm. Find the image.
- What is the power of a lens with $f = 25$ cm?

Answers: $s' = -10$ cm, $M = +\frac{1}{2}$. $f = 20$ cm, so $s' = 60$ cm and $M = -2$: real, inverted, twice the size. 4.0 dioptres.

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Thermal transport, calorimetry and expansion

GRE Subject Test · GRE Physics

Around 1761 Joseph Black, a professor in Glasgow, put a flask of ice and a flask of ice-cold water side by side in a warm room. The water warmed by several degrees in half an hour. The ice stayed at its melting point for more than ten hours while it slowly melted, though it was taking in heat all the time. Black called this hidden heat **latent heat** 潜热. His careful heat bookkeeping, **calorimetry** 量热法, is still how we solve mixing problems.

GRE Physics thermal questions ask you to:

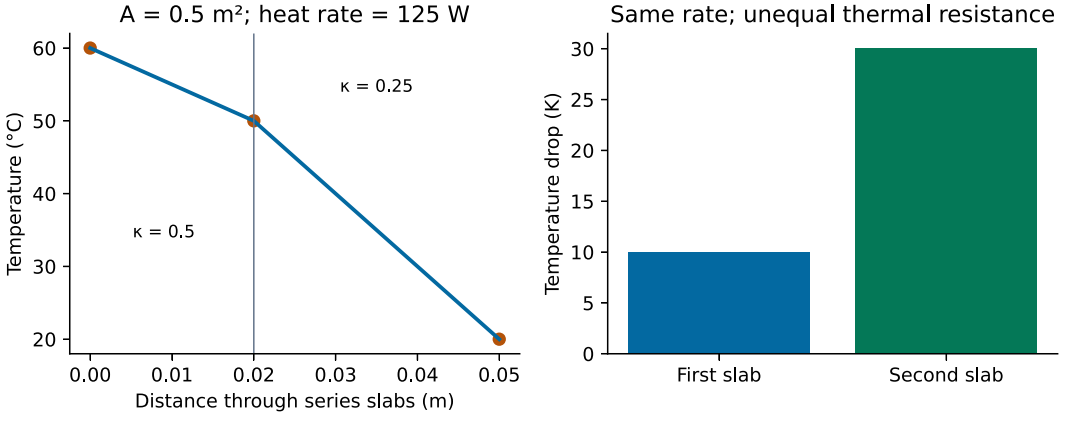
- Use **conduction** 热传导 and **thermal resistance** 热阻 through layers.
- Balance heat in mixing problems with **specific heat capacity** 比热容 and latent heat.
- Use **thermal expansion** 热膨胀 and **thermal stress** 热应力.
- Use radiation and Newton's law of cooling.

Conduction and thermal resistance

Heat flows through a slab of thickness L, area A and conductivity κ at the rate:

$$\dot{Q} = \frac{\kappa A\,\Delta T}{L} = \frac{\Delta T}{R_{\text{th}}}, \qquad R_{\text{th}} = \frac{L}{\kappa A}$$

- Layers in series carry the same $\dot{Q}$, and their resistances add, like resistors in series.
- Each layer's temperature drop is $\dot{Q}R_{\text{th}}$, so the layer with more resistance takes the bigger drop.
- Parallel paths share the same ΔT, and their heat rates add.



The same heat current crosses both slabs; the slab with more thermal resistance takes the larger drop

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Calorimetry and latent heat

In an isolated system, the heat gained equals the heat lost: $\sum mc\,\Delta T = 0$, with each ΔT taken as final minus initial.

- Include the container if its heat capacity is given.
- At a phase change, heat $Q = m\ell$ changes the phase, not the temperature.
- First check whether there is enough heat to complete the phase change; if not, the final state is a mixture at the transition temperature.

Expansion and thermal stress

$$\Delta L = \alpha L\,\Delta T$$

- Areas expand by 2α and volumes by 3α (for small changes).
- A hole in a plate grows when the plate is heated, as if it were made of the same material.
- A rod held between fixed walls cannot expand, so it feels a stress $-Y\alpha\,\Delta T$, where Y is Young's modulus.

Radiation and cooling

- Net radiated power: $\varepsilon\sigma A(T^4 - T_{\text{env}}^4)$, with temperatures in kelvin. Doubling the absolute temperature gives 16 times the emission.
- Newton's law of cooling: for small temperature differences, the excess $T - T_{\text{env}}$ decays exponentially with time constant $C/(hA)$.

Worked example: a two-layer wall

A wall of area 0.50 m^2 has two layers: 0.020 m thick with $\kappa = 0.50$ W/(m K), and 0.030 m thick with $\kappa = 0.25$ W/(m K). Inside is 60 °C, outside 20 °C.

- $R_1 = 0.020/(0.50 \times 0.50) = 0.080$ K/W and $R_2 = 0.030/(0.25 \times 0.50) = 0.24$ K/W.
- $\dot{Q} = 40/0.32 = 125$ W.
- Drops: $125 \times 0.080 = 10$ K and $125 \times 0.24 = 30$ K, so the interface is at 50 °C.

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Worked example: ice in warm water

0.20 kg of ice at 0 °C is dropped into 1.0 kg of water at 40 °C. Take $c = 4.18$ kJ/(kg K) and $\ell = 334$ kJ/kg.

- Melting all the ice needs $0.20 \times 334 = 67$ kJ; cooling the water to 0 °C would release $1.0 \times 4.18 \times 40 = 167$ kJ. So all the ice melts.
- Balance: $1.0(4.18)(40 - T) = 66.8 + 0.20(4.18)T$, so $T = 20$ °C.
- A 2.0 m steel rod ($\alpha = 1.0 \times 10^{-5}$ K^{-1}) heated by 50 K grows 1.0 mm; clamped, with $Y = 100$ GPa, it feels 50 MPa of compression.

Common errors

- Giving both layers of a wall the same temperature drop.
- Ignoring the latent heat, or letting the temperature change during a phase change.
- Thinking a hole shrinks when a plate is heated.
- Using degrees Celsius in the radiation law.
- Applying Newton's law of cooling to the temperature instead of the excess over the surroundings.

Key terms

latent heat: Heat that changes a phase without changing the temperature.

calorimetry: Measuring heat by balancing the heat gained and lost in an isolated system.

conduction: Heat flow through a material from hot to cold, without the material moving.

thermal resistance: $L/(\kappa A)$, the temperature drop per watt of heat flow.

specific heat capacity: The heat needed to warm 1 kg of a substance by 1 K.

thermal expansion: The growth of an object's size with temperature.

thermal stress: The stress in an object that is stopped from expanding or contracting.

Check yourself

- 2.0 kg of water at 80 °C is mixed with 1.0 kg at 20 °C. What is the final temperature?
- A body's absolute temperature doubles. By what factor does its radiated power grow?
- A metal plate with a hole in it is heated. What happens to the hole?

Answers: 60 °C. 16. It gets bigger.

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Partition derivatives, energy fluctuations and heat capacity

GRE Subject Test · GRE Physics

In 1902 Josiah Willard Gibbs, at Yale, published *Elementary Principles in Statistical Mechanics*. He imagined a huge collection of copies of one system, all at the same temperature, and asked how often each state appears. From one function of temperature, now called the **partition function** 配分函数, he could get the energy, the entropy, the **free energy** 自由能 and the **heat capacity** 热容. Almost every GRE statistical mechanics question is a derivative of Z.

GRE Physics questions on the canonical ensemble ask you to:

- Build Z with degeneracies and find level probabilities.
- Get U, F, S and C_V from Z.
- Relate heat capacity to energy **fluctuations** 涨落.
- Analyse the **two-level system** 二能级系统 and its **Schottky peak** 肖特基峰.

The partition function

With $\beta = 1/k_B T$:

$$Z = \sum_i g_i \, e^{-\beta E_i}, \qquad p_i = \frac{g_i \, e^{-\beta E_i}}{Z}, \qquad U = -\frac{\partial \ln Z}{\partial \beta}$$

- Differentiate with the energy levels held fixed.
- For N independent, distinguishable subsystems, $Z_{\text{tot}} = z^N$ and $U_{\text{tot}} = Nu$.

Free energy, entropy and heat capacity

$$F = -k_B T \ln Z, \qquad S = k_B(\ln Z + \beta U), \qquad C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

- Shifting every level by the same constant shifts U and F by that constant, but leaves the probabilities, S and C_V unchanged.

Energy fluctuations

The energy of a system in contact with a heat bath fluctuates. Its variance is tied to the heat capacity:

$$C_V = \frac{\mathrm{Var}(E)}{k_B T^2} = \frac{\langle E^2 \rangle - U^2}{k_B T^2}$$

- So C_V can never be negative.

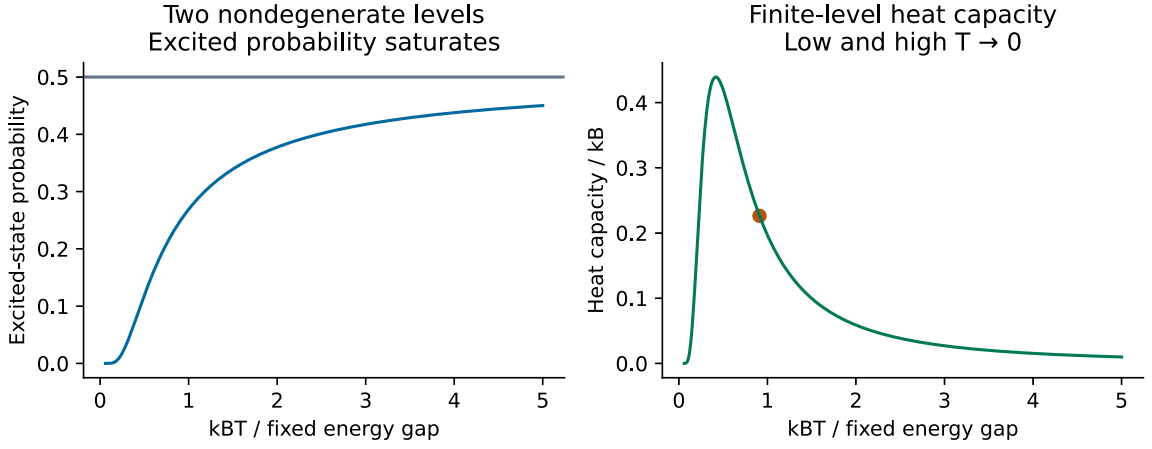
- For N subsystems, the relative fluctuation $\sqrt{\mathrm{Var}(E)}/U$ shrinks as $1/\sqrt{N}$, which is why large systems have sharp energies.

The two-level system

Levels at 0 and Δ, each with one state. Write $x = \Delta/k_BT$:

$$p_{\mathrm{exc}} = \frac{1}{1+e^{x}}, \qquad \frac{C}{k_B} = \frac{x^2 e^{x}}{(1+e^{x})^2}$$

- At low temperature almost nothing is excited, so little heat is absorbed: $C \to 0$.
- At high temperature both levels are almost equally filled: $p_{\mathrm{exc}} \to \tfrac{1}{2}$, $U \to \Delta/2$, $S \to k_B \ln 2$, and again $C \to 0$.
- In between, C has a single peak near $k_BT \approx 0.42\Delta$: the Schottky peak.



A finite level system has a Schottky peak; its heat capacity vanishes at low and high T

Worked example: a two-level system at kT = Δ/ln 3

Levels 0 and Δ, with $\Delta/k_BT = \ln 3$, so $e^{-x} = \tfrac{1}{3}$.

- $z = 1 + \tfrac{1}{3} = \tfrac{4}{3}$, so $p_{\mathrm{exc}} = \tfrac{1}{4}$ and $U = \Delta/4$.
- $\langle E^2 \rangle = \Delta^2/4$, so $\mathrm{Var}(E) = \Delta^2/4 - \Delta^2/16 = \tfrac{3}{16}\Delta^2$.
- $C/k_B = \mathrm{Var}(E)/(k_BT)^2 = \tfrac{3}{16}(\ln 3)^2 = 0.23$, the same as the formula above gives.

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Worked example: spins in a magnetic field

A paramagnetic salt has two-level ions with $\Delta = 1.0$ meV.

- The Schottky peak is near $T = 0.42\Delta/k_B = 0.42\times(1.0\times10^{-3})/(8.62\times10^{-5}) = 4.9$ K.
- At high temperature the entropy per mole approaches $R\ln 2 = 5.8$ J/K.
- Magnetic cooling works by changing Δ with the field while the entropy stays fixed.

Common errors

- Forgetting the degeneracy g_i in Z.
- Differentiating with respect to T when the formula uses β, without the chain rule.
- Thinking a finite level system's heat capacity grows without limit at high temperature.
- Thinking that shifting all levels changes the entropy.
- Allowing a negative heat capacity in the canonical ensemble.

Key terms

partition function: $Z = \sum g_i e^{-E_i/k_BT}$, the sum that normalises the probabilities.

free energy: $F = U - TS = -k_BT\ln Z$.

heat capacity: The heat needed per unit rise in temperature.

fluctuation: Random variation of a quantity about its mean.

two-level system: A system with only two energy levels.

Schottky peak: The heat-capacity peak of a system with a finite set of levels.

Check yourself

- Levels 0 ($g=1$) and ε ($g=2$) at $k_BT = \varepsilon/\ln 2$. What is Z?
- Why can the heat capacity in the canonical ensemble never be negative?
- Every level is shifted up by 5 meV. Does the entropy change?

Answers: $1 + 2\times\tfrac{1}{2} = 2$. Because $C_V = \mathrm{Var}(E)/k_BT^2$ and a variance is never negative. No.

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Quantum occupations and limits of equipartition

GRE Subject Test · GRE Physics

Quantum occupations and the limits of equipartition

In 1875 Heinrich Weber measured the heat capacity of diamond and found it was only about a quarter of the classical value, 3R per mole, that most solids reach at room temperature. On heating, it rose towards 3R. In 1907 Albert Einstein explained this: if each atom vibrates like a quantum oscillator, a vibration whose energy step is much bigger than kT cannot take up heat. **Equipartition** 能量均分 fails when the levels are too far apart, and the vibrations **freeze out** 冻结.

GRE Physics quantum-statistics questions ask you to:

- Use fermion and boson **occupation numbers** 占据数 with the **chemical potential** 化学势.
- Count arrangements of **identical particles** 全同粒子.
- Recognise the classical limit.
- Know when equipartition holds and when it fails, as in the **Einstein solid** 爱因斯坦固体.

Mean occupations

For a single complete state (spin included) of energy ε, with $x = (\varepsilon - \mu)/k_B T$:

$$\bar{n}_{\mathrm{FD}} = \frac{1}{e^{x} + 1}, \qquad \bar{n}_{\mathrm{BE}} = \frac{1}{e^{x} - 1}$$

- Fermions: between 0 and 1. Bosons: can be larger than 1.
- A level with g states holds $g\bar{n}$ on average. A level with five states can hold up to five fermions.
- Photons have $\mu = 0$ because their number is not conserved.

Counting two particles

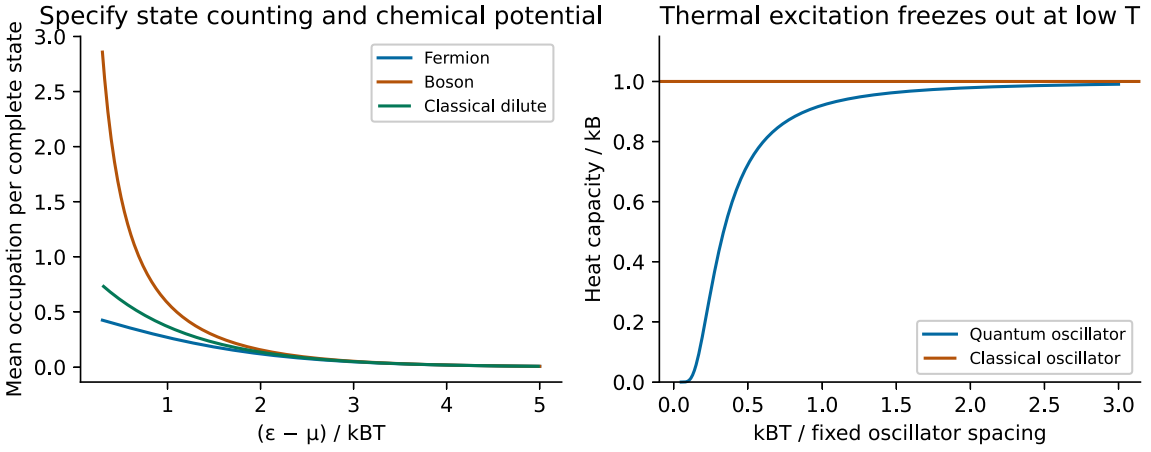
Put two particles into 4 complete states:

- Two identical fermions: $\binom{4}{2} = 6$ ways (no state used twice).
- Two identical bosons: $\binom{5}{2} = 10$ ways (sharing allowed).
- Two labelled (distinguishable) particles: $4^2 = 16$ ways.
- Two electrons with opposite spins can share one spatial orbital, because they are in different complete states.

The classical limit

When x is large, both occupations tend to e^{-x}, the Maxwell–Boltzmann result.

- At $x = \ln 4$: fermions $\frac{1}{5}$, bosons $\frac{1}{3}$, classical $\frac{1}{4}$, still clearly different.
- Fermions at low temperature fill every state below μ and none above, with $\bar{n} = \frac{1}{2}$ at $\varepsilon = \mu$.



All three distributions agree when $(\varepsilon - \mu) \gg kT$; quantum heat capacity freezes out at low T

Equipartition and its limits

Classically, each quadratic term in the energy contributes $\frac{1}{2}k_BT$.

- Monatomic gas: $C_V = \frac{3}{2}Nk_B$. One oscillator (kinetic plus potential): $C = k_B$.
- A quantum oscillator with $x = \hbar\omega/k_BT$ has $\dfrac{C}{k_B} = \dfrac{x^2e^x}{(e^x-1)^2}$.
- When $k_BT \ll \hbar\omega$, $C \to 0$: molecular vibrations at room temperature add almost nothing to the heat capacity of air.

The Einstein solid

Einstein modelled a solid as $3N$ identical oscillators of frequency ω.

- At high temperature, $C \to 3Nk_B = 3R$ per mole (the Dulong–Petit law).
- At low temperature, C falls towards zero. Diamond's stiff bonds give a high frequency, so it is still frozen out at room temperature.

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Worked example: occupations and counting

- At $x = \ln 4$: $\bar{n}_{\mathrm{FD}} = 1/(4+1) = 0.20$; $\bar{n}_{\mathrm{BE}} = 1/(4-1) = 0.33$; classical $e^{-x} = 0.25$.
- A quantum oscillator at $x = \ln 2$: mean excitation $\hbar\omega/(e^x - 1) = \hbar\omega$, and $C/k_B = 2(\ln 2)^2 = 0.96$, just below the classical 1.
- Two identical fermions in 4 states: 6 arrangements, against 16 for labelled particles.

Worked example: diamond at room temperature

Take diamond's Einstein temperature $\hbar\omega/k_B = 1320$ K, at 300 K.

- $x = 1320/300 = 4.4$ and $e^x = 81.5$.
- $C/k_B = \dfrac{(4.4)^2(81.5)}{(80.5)^2} = 0.24$ per oscillator, so $C = 0.24 \times 3R = 6.1$ J/(mol K).
- This matches the measured value, about a quarter of the classical 24.9 J/(mol K).

Common errors

- Applying the Pauli limit per energy level instead of per complete state.
- Using $\mu \neq 0$ for photons.
- Counting identical particles as if they were labelled.
- Using equipartition for vibrations whose spacing is much larger than $k_B T$.
- Forgetting the factor 3 (three directions) in the Einstein solid.

Key terms

equipartition: Each quadratic energy term has an average of $\frac{1}{2}k_B T$ in classical equilibrium.

freeze out: A degree of freedom stops taking up heat when $k_B T$ is much smaller than its level spacing.

occupation number: The mean number of particles in one quantum state.

chemical potential: The energy cost of adding one particle; it sets the occupations.

identical particles: Particles that cannot be told apart, even in principle.

Einstein solid: A model of a solid as identical quantum oscillators.

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Check yourself

- Two identical bosons share 3 states. How many arrangements are there? And for fermions?
- What is the fermion occupation at $\varepsilon = \mu$?
- What is the heat capacity of one quantum oscillator at very high temperature?

Answers: 6 for bosons, 3 for fermions. $\frac{1}{2}$. k_B, the classical value.

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Weak perturbations and degenerate subspaces

GRE Subject Test · GRE Physics

In 1913 Johannes Stark put hydrogen atoms in a very strong electric field and found that each Balmer line split into several components. Bohr's model could not fully explain the pattern. In 1926 Erwin Schrödinger treated the field as a small extra term, a **perturbation** 微扰, added to the hydrogen problem he had already solved, and his result matched Stark's measurements. Perturbation theory is how physicists handle almost every real system that is "nearly" a solved one.

GRE Physics perturbation questions ask you to:

- Find the **first-order shift** 一级修正 as an **expectation value** 期望值.
- Use symmetry to show a first-order shift is zero.
- Know the sign and size of the **second-order shift** 二级修正.
- Diagonalise the perturbation in a **degenerate** 简并的 subspace.

First order

Write $H = H_0 + \lambda W$, where H_0 has known states $|n\rangle$. To first order:

$$E_n^{(1)} = \lambda \langle n|W|n\rangle = \lambda \int |\psi_n|^2 \, W \, dx$$

- The shift is W averaged over the unperturbed probability density.
- Infinite well $0 < x < L$: every state has $\langle x\rangle = L/2$, so $W = \gamma x$ shifts every level by $\gamma L/2$.

Symmetry

If $|\psi_n|^2$ is symmetric about a point and W is odd about it, the first-order shift is zero.

- In a symmetric well, a tilt $\gamma(x - L/2)$ gives zero first-order shift for every state, though the wavefunctions do change.
- An even perturbation such as $\eta(x - L/2)^2$ gives a positive shift that depends on n.

Second order

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\lambda W_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

- For the ground state every denominator is negative, so $E^{(2)} \leq 0$: the ground state is always pushed down.

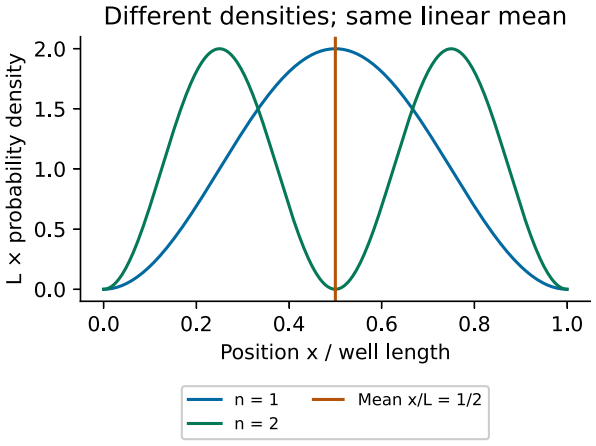
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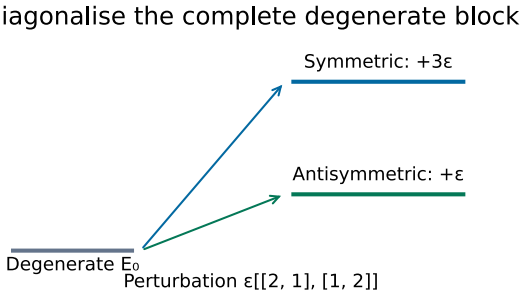
- The method works only when each coupling $|\lambda W_{mn}|$ is small compared with the gap $|E_n - E_m|$; nearly degenerate levels break it.

Degenerate levels

When several states share an energy, write W as a matrix in that subspace and diagonalise it. The eigenvalues are the first-order shifts, and the eigenvectors are the correct starting states.

- Reading only the diagonal elements gives the wrong answer.
- A perturbation that is a multiple of the identity in the subspace shifts all the states equally and never splits the level.
- This is how the Stark effect splits the degenerate $n = 2$ level of hydrogen.





Diagonalise the whole degenerate block: the eigenvalues give the first-order shifts

Worked example: a tilted well

An electron in an infinite well of width 2.0 nm feels $W = \gamma x$ with $\gamma = 0.050$ eV/nm.

- $\langle x \rangle = 1.0$ nm for every state, so every level rises by $0.050 \times 1.0 = 0.050$ eV.
- Measure the tilt from the centre instead, $\gamma(x - L/2)$: the first-order shift is zero for every state.
- The two answers differ only by the constant $\gamma L/2$, which shifts all levels equally.

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Worked example: a degenerate pair

Two degenerate states feel $\varepsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- Eigenvalues: 3ε, for $(1, 1)/\sqrt{2}$, and ε, for $(1, -1)/\sqrt{2}$.
- Reading the diagonal would give 2ε twice, with no splitting.
- A harmonic oscillator with $W = \lambda x$ has zero first-order shift (odd perturbation); the exact shift $-\lambda^2/(2m\omega^2)$ is second order and negative, as the rule says.

Common errors

- Using the diagonal elements alone in a degenerate subspace.
- Forgetting that odd perturbations give zero first-order shifts in symmetric states.
- Giving the ground state a positive second-order shift.
- Applying perturbation theory when the coupling is as large as the level gap.
- Thinking a constant perturbation can split a degenerate level.

Key terms

perturbation: A small extra term added to a Hamiltonian that has already been solved.

first-order shift: The energy change $\langle n|W|n\rangle$, the perturbation averaged over the unperturbed state.

expectation value: The average result of measuring a quantity over many identical preparations.

second-order shift: The energy change from couplings to other states, divided by the energy gaps.

degenerate: Sharing the same energy with another independent state.

Check yourself

- A perturbation is a constant c. What is the first-order shift of every level?
- A degenerate pair feels εI, a multiple of the identity. Does the level split?
- What is the sign of the second-order shift of a ground state?

Answers: c. No; both states rise by ε. Negative or zero.

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Exchange symmetry, spin pairs and Pauli exclusion

GRE Subject Test · GRE Physics

In 1927 Walter Heitler and Fritz London calculated the energy of two hydrogen atoms brought close together. With the electrons' spins opposite (a **singlet** 单态), the energy fell to a minimum: the atoms formed a molecule. With the spins parallel (a **triplet** 三重态), the atoms repelled. No new force was involved. The difference came from how the wavefunction of two **identical particles** 全同粒子 must behave when they swap places. This was the first quantum explanation of the chemical bond.

GRE Physics questions on identical particles ask you to:

- Apply **exchange symmetry** 交换对称性 to the total state, space and spin together.
- Build singlets and triplets.
- Count two-particle states with the **Pauli exclusion principle** 泡利不相容原理.
- Explain exchange correlation: why identical fermions avoid each other.

Exchange symmetry

Swapping all the labels of two identical particles (position and spin) changes the sign of the state for fermions and leaves it unchanged for bosons.

- With two different orbitals a and b: $\Psi_{\pm} = \frac{1}{\sqrt{2}}\,[a(1)b(2) \pm b(1)a(2)]$.
- If $a = b$, only $a(1)a(2)$ exists, and it is symmetric.
- For electrons the total state is **antisymmetric** 反对称, so two electrons can never be in the same complete state.

Singlet and triplet

Two spin-½ particles have four spin states.

- Singlet: $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$, total spin $S = 0$, antisymmetric.
- Triplet: $\uparrow\uparrow$, $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$, $\downarrow\downarrow$, total spin $S = 1$, symmetric.
- For electrons: symmetric space goes with the singlet, antisymmetric space with the triplet.

Counting states

Count complete states (orbital and spin) first, then choose.

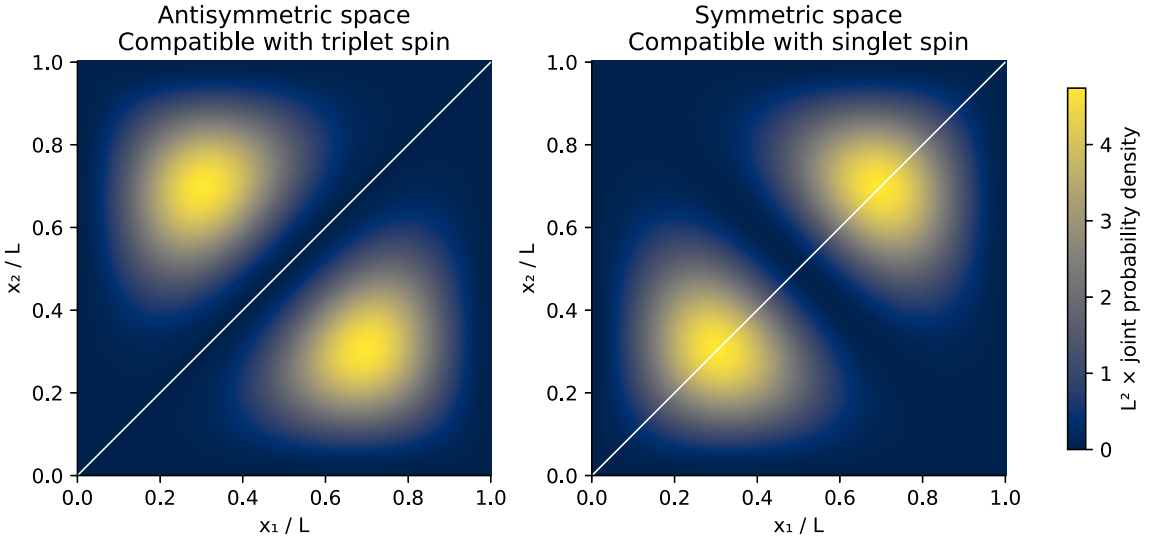
- Two orbitals × two spins = 4 complete states, so two electrons have $\binom{4}{2} = 6$ states: two singlets with both electrons in one orbital, one open-shell singlet, and three triplets.

- Three orbitals: $\binom{6}{2} = 15$ two-electron states.
- Two spinless bosons in two orbitals: 3 states (aa, bb and the symmetric ab).

Exchange correlation

An antisymmetric spatial state vanishes when the two particles are at the same place: $\Psi_-(x, x) = 0$.

- Particles in Ψ_- keep apart, and those in Ψ_+ bunch together, though no force acts.
- In helium, the triplet ("orthohelium") states have the electrons further apart, so they feel less repulsion and lie lower than the matching singlets.



An antisymmetric space wavefunction pairs with triplet spin, a symmetric one with singlet spin

Worked example: two particles in a well

Use the infinite-well states $a = \sqrt{2/L}\sin(\pi x/L)$ and $b = \sqrt{2/L}\sin(2\pi x/L)$, at $x_1 = L/4$ and $x_2 = 3L/4$.

- $a(L/4) = a(3L/4) = 1/\sqrt{L}$; $b(L/4) = +\sqrt{2/L}$ and $b(3L/4) = -\sqrt{2/L}$.
- $\Psi_- = \frac{1}{\sqrt{2}}\left[-\frac{\sqrt{2}}{L} - \frac{\sqrt{2}}{L}\right] = -\frac{2}{L}$, a density of $4/L^2$.
- $\Psi_+ = 0$ at these points: the symmetric state puts no weight on this pair of positions.

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Worked example: counting

- Two electrons in 3 orbitals: 6 complete states, so $\binom{6}{2} = 15$ states. Of these, $3 \times 3 = 9$ are triplets (3 antisymmetric orbital pairs $\times$ 3 spin states) and 6 are singlets (6 symmetric orbital pairs).
- Two electrons in one orbital: only the singlet.
- Two labelled particles in 4 complete states would have $4^2 = 16$ states.

Common errors

- Requiring the spatial part alone to be antisymmetric for electrons.
- Putting two electrons with parallel spins in one orbital.
- Counting identical particles as if they were labelled.
- Thinking the singlet and triplet differ in energy because of a magnetic force; the main effect is exchange.
- Forgetting that the symmetric ab state counts for bosons.

Key terms

singlet: The two-particle spin state with total spin 0, antisymmetric under exchange.

triplet: The three two-particle spin states with total spin 1, symmetric under exchange.

identical particles: Particles that cannot be told apart, even in principle.

exchange symmetry: How a many-particle state behaves when two particles swap labels.

Pauli exclusion principle: No two identical fermions can occupy the same complete quantum state.

antisymmetric: Changing sign when two labels are swapped.

Check yourself

- How many spin states do two spin-½ particles have, and how are they split?
- Two electrons share one spatial orbital. What is their spin state?
- How many states do two spinless bosons have in two orbitals?

Answers: 4: one singlet and three triplets. The singlet. 3.

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Finite-well bound states and hydrogenic quantum numbers

GRE Subject Test · GRE Physics

In 1928 George Gamow, and separately Ronald Gurney and Edward Condon, explained a puzzle of radioactivity. An alpha particle inside a nucleus does not have enough energy to climb over the barrier holding it in, yet sometimes it escapes. In quantum mechanics a particle in a **finite square well** 有限深势阱 has a wavefunction that leaks into the walls and decays there, but is not zero. If the wall is thin enough, there is a small chance of **tunnelling** 隧穿 out. The calculation gave the right link between alpha energies and half-lives.

GRE Physics questions on wells and hydrogen ask you to:

- Match wavefunctions at the walls of a finite well.
- Solve for **bound state** 束缚态 energies, at least graphically.
- Tell bound states from **scattering states** 散射态.
- Use hydrogen's quantum numbers, degeneracy and **radial probability** 径向概率.

The finite square well

Take $V = -V_0$ for $|x| < a$ and $V = 0$ outside. Bound states have $-V_0 < E < 0$.

- Inside: $k = \sqrt{2m(E + V_0)}/\hbar$, and the wavefunction oscillates.
- Outside: the **decay constant** 衰减常数 $\kappa = \sqrt{-2mE}/\hbar$, and the wavefunction falls as $e^{-\kappa|x|}$.
- Match both ψ and $d\psi/dx$ at each wall. Even states need $k \tan ka = \kappa$; odd states need $-k \cot ka = \kappa$.

Solving for the energies

With $z = ka$ and $\rho = a\sqrt{2mV_0}/\hbar$, the even states satisfy:

$$z \tan z = \sqrt{\rho^2 - z^2}, \qquad \frac{E}{V_0} = \frac{z^2}{\rho^2} - 1$$

- Solve graphically or numerically.
- In one dimension every attractive well has at least one bound state; when $\rho < \pi/2$ there is only that one, and no odd state.

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Bound and scattering states

- Bound states: discrete energies, normalisable wavefunctions, decaying tails.
- Scattering states ($E > 0$): any energy, travelling waves outside.
- The decaying tail is what makes tunnelling through a finite barrier possible. Find the probability in the tail by integrating $|\psi|^2$ after normalising; the raw amplitude is not a probability.

Hydrogen-like quantum numbers

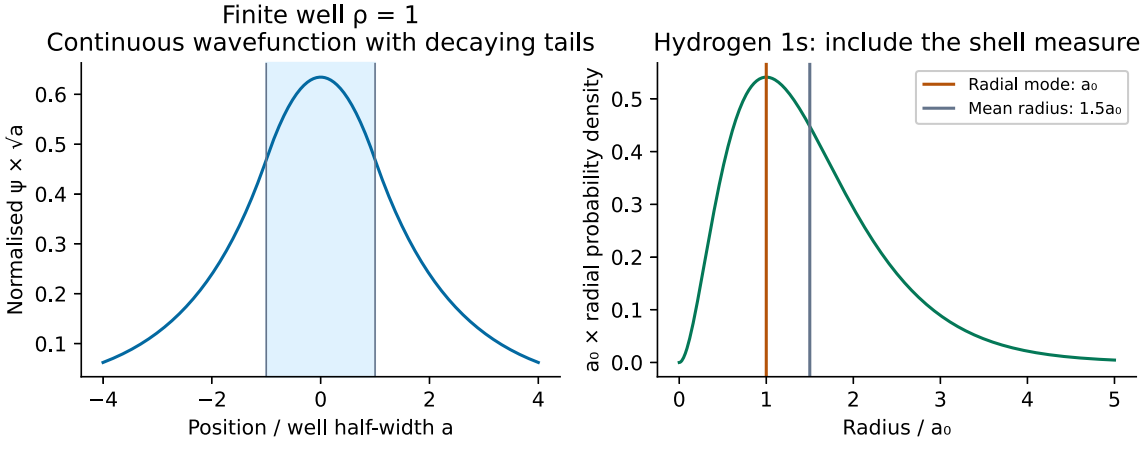
- $E_n = -13.6\,Z^2/n^2$ eV; $l = 0, \ldots, n-1$; $L^2 = l(l+1)\hbar^2$.
- Each n has n^2 spatial states, and $2n^2$ with spin.
- m_l runs from $-l$ to $+l$, and $L_z = m_l\hbar$.

Radial probability

The probability of finding the electron between r and $r + dr$ includes the area of the shell, $4\pi r^2$:

$$P(r) = r^2|R(r)|^2 \quad \text{(normalised)}$$

- For 1s, $|\psi|^2$ is largest at the nucleus, but $P(r)$ peaks at the Bohr radius a_0.
- The mean radius of 1s is $\frac{3}{2}a_0$.



Include the r^2 shell factor: the most probable radius is a_0, the mean $1.5a_0$

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Worked example: a shallow well

Take $\rho = 1$.

- The even condition becomes $z \tan z = \sqrt{1 - z^2}$, which simplifies to $z = \cos z$. So $z = 0.739$.
- $E/V_0 = 0.739^2 - 1 = -0.454$.
- Since $\rho < \pi/2$, this is the only bound state.

Worked example: hydrogen 1s and n = 3

- For 1s, $P(r) = \dfrac{4r^2}{a_0^3} e^{-2r/a_0}$. Setting $dP/dr = 0$ gives $r = a_0$.
- The $n = 3$ level has $l = 0, 1, 2$ and $1 + 3 + 5 = 9$ spatial states, or 18 with spin.
- A $3d$ electron ($l = 2$) has $L^2 = 6\hbar^2$, so $|L| = \sqrt{6}\,\hbar$, never $2\hbar$.

Common errors

- Matching ψ but not $d\psi/dx$ at a finite wall.
- Saying a shallow one-dimensional well may have no bound state.
- Reading the tail amplitude as a probability.
- Putting the most probable 1s radius at the nucleus; the r^2 factor moves it to a_0.
- Writing $|L| = l\hbar$ instead of $\sqrt{l(l+1)}\,\hbar$.

Key terms

finite square well: A potential well with walls of finite height.

tunnelling: Passing through a region where the energy is below the potential.

bound state: A state confined to a region, with discrete energy.

scattering state: An unbound state with any energy above the potential outside.

decay constant: κ, the rate at which a wavefunction falls off inside a barrier.

radial probability: The probability per unit radius of finding the electron, including the shell factor r^2.

Check yourself

- A one-dimensional well has $\rho < \pi/2$. How many bound states does it have?
- What is the most probable radius of a hydrogen 1s electron?
- How many spatial states does hydrogen's $n = 4$ level have?

Answers: one, an even state. a_0. 16.

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Atomic field shifts and spectral differences

GRE Subject Test · GRE Physics

In 1896 Pieter Zeeman, in Leiden, held a sodium flame between the poles of a strong electromagnet and looked at the yellow D lines through a fine diffraction grating. When he switched the magnet on, the lines became wider. Hendrik Lorentz explained the effect with oscillating electrons and, from the size of the widening, worked out the charge-to-mass ratio of the radiating particle a year before J. J. Thomson measured it for the electron. Better instruments later showed the lines splitting into several sharp components: the **Zeeman effect** 塞曼效应.

GRE Physics questions on atoms in fields ask you to:

- Use the strong-field (**Paschen–Back effect** 帕邢 – 巴克效应) and weak-field Zeeman formulas.
- Calculate the **Landé g-factor** 朗德 g 因子.
- Count spectral lines from level shifts.
- Know when the **Stark effect** 斯塔克效应 is linear.

Strong field: uncoupled labels

When the magnetic energy is much larger than the **spin–orbit coupling** 自旋 – 轨道耦合, L and S act separately:

$$\Delta E = \mu_B B\,(m_l + 2m_s), \qquad \mu_B = 5.79 \times 10^{-5}\ \text{eV/T}$$

- μ_B is the **Bohr magneton** 玻尔磁子. The factor 2 on m_s is the electron spin g-factor.

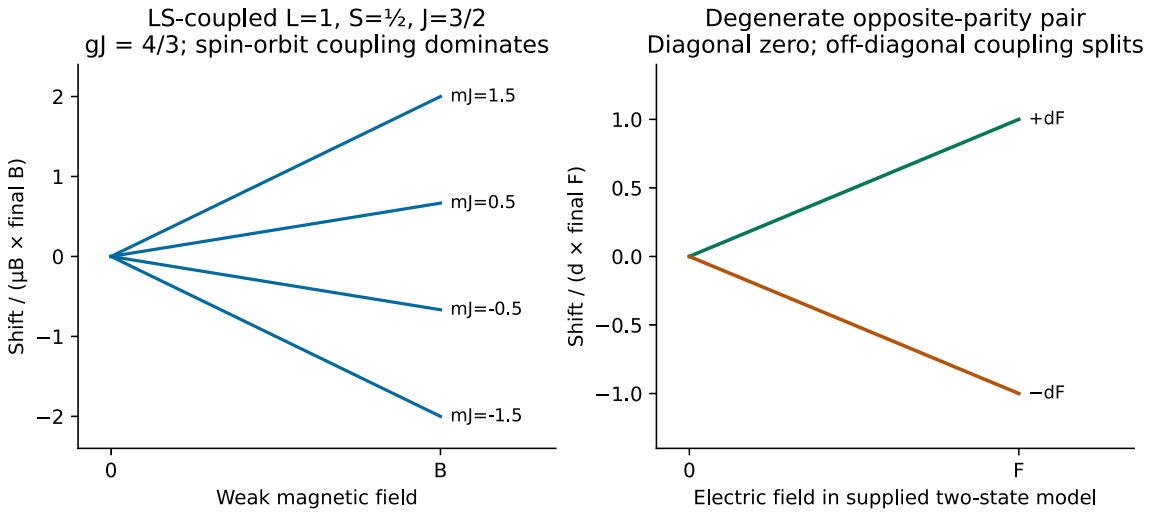
Weak field: the Landé g-factor

When the field is weak compared with the spin–orbit splitting, J stays a good quantum number:

$$\Delta E = g_J\,\mu_B B\,m_J, \qquad g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

- For $L = 1$, $S = \tfrac{1}{2}$: $J = \tfrac{3}{2}$ has $g_J = \tfrac{4}{3}$, and $J = \tfrac{1}{2}$ has $g_J = \tfrac{2}{3}$.
- Each J level splits into $2J + 1$ equally spaced sublevels.

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Weak-field Zeeman shifts are $gJ \cdot mJ \cdot \mu B \cdot B$; an electric field splits a degenerate opposite-parity pair linearly

Counting lines

A photon's shift is the upper level's shift minus the lower level's.

- Normal Zeeman effect ($L = 1 \to 0$, spin ignored): $\Delta m_l = 0, \pm 1$ gives three lines, at $-\mu_B B$, 0 and $+\mu_B B$.
- With spin and different g_J in the two levels, the anomalous effect gives more lines.
- Several transitions can share one frequency, so count distinct lines, not levels.

The Stark effect

An electric field F adds eFz.

- A non-degenerate state of definite parity has $\langle z \rangle = 0$: no linear shift, only a smaller quadratic one.
- A degenerate pair of opposite parity, with dipole matrix $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$, shifts by $\pm dF$: a linear splitting of $2|dF|$. Hydrogen's $n = 2$ level behaves like this.

Which regime?

Compare $\mu_B B$ with the fine-structure splitting.

- Sodium's 3p splitting is 2.1 meV, so $\mu_B B$ matches it only near $B = 36$ T. Laboratory fields of about 1 T are therefore weak.
- In hydrogen, whose fine structure is much smaller, a few tesla already approach the strong-field regime.

Worked example: weak-field shifts

A $J = \frac{3}{2}$ level ($g_J = \frac{4}{3}$) sits in $B = 2.0$ T.

- $m_J = \frac{1}{2}$: $\Delta E = \frac{4}{3} \times \frac{1}{2} \times (5.79 \times 10^{-5}) \times 2.0 = 7.7 \times 10^{-5}$ eV.
- Neighbouring sublevels are $\frac{4}{3}(5.79 \times 10^{-5})(2.0) = 1.5 \times 10^{-4}$ eV apart.
- Four sublevels, at $m_J = \pm\frac{3}{2}, \pm\frac{1}{2}$.

Worked example: the normal triplet

A line at 500 nm (2.48 eV) shows the normal Zeeman effect in 1.0 T.

- The outer lines shift by $\pm\mu_B B = \pm 5.79 \times 10^{-5}$ eV.
- $\Delta\lambda = \lambda\,\Delta E/E = 500 \times (5.79 \times 10^{-5})/2.48 = 0.012$ nm.
- That is why Zeeman needed a fine grating, and saw only a widening at first.

Common errors

- Using $g_J m_J$ in a strong field, or $m_l + 2m_s$ in a weak one.
- Forgetting the factor 2 on the spin.
- Giving a photon the shift of one level instead of the difference.
- Expecting a linear Stark shift for a non-degenerate state.
- Assuming a 1 T laboratory field is strong for every atom.

Key terms

Zeeman effect: The splitting of spectral lines in a magnetic field.

Paschen–Back effect: The strong-field Zeeman regime, where L and S act separately.

Landé g-factor: g_J, which sets the size of weak-field Zeeman shifts for a level with given L, S, J.

Stark effect: The shift and splitting of energy levels in an electric field.

spin–orbit coupling: The interaction between an electron's spin and its orbital motion.

Bohr magneton: $\mu_B = e\hbar/2m_e$, the natural unit of atomic magnetic moment.

Check yourself

- What is g_J for $L = 1$, $S = \frac{1}{2}$, $J = \frac{1}{2}$?
- Does the ground state of hydrogen have a linear Stark shift?
- Into how many sublevels does a $J = \frac{5}{2}$ level split in a weak field?

Answers: $\frac{2}{3}$. No: it is non-degenerate with definite parity. Six.

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Thermal spectra and one-electron scaling

GRE Subject Test · GRE Physics

Blackbody spectra and one-electron atoms

In 1896 Edward Pickering at Harvard found a series of lines in the spectrum of the hot star ζ Puppis. They looked like hydrogen's Balmer lines, but some fell between them, and astronomers called them "hydrogen" for years. In 1913 Niels Bohr showed they came from ionised helium, He^+: one electron, but a nucleus of charge $Z = 2$. Every level sits $Z^2 = 4$ times deeper than in hydrogen. Laboratory tests that year confirmed it, a strong early success for Bohr's model.

GRE Physics questions on thermal radiation and simple atoms ask you to:

- Use **blackbody** 黑体 radiation: Planck's law, Wien's law and the Stefan–Boltzmann law.
- Find the power radiated by a grey body with **emissivity** 发射率 ε.
- Scale Bohr-model radii, speeds and energies with n and Z.
- Calculate line energies, the **series limit** 谱线系限 and ionisation energies.

Planck's law

A blackbody absorbs all radiation that falls on it and emits a spectrum that depends only on its temperature. Planck's law gives the **spectral radiance** 光谱辐亮度 per unit wavelength:

$$B_\lambda = \frac{2hc^2}{\lambda^5}\,\frac{1}{e^{hc/\lambda k_B T} - 1}$$

- Wien's law: $\lambda_{\max} T = 2.898 \times 10^{-3}$ m · K. Doubling T halves the peak wavelength.
- Stefan–Boltzmann law: total flux from the surface $= \sigma T^4$, with $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.
- The peak of the per-frequency curve B_ν is not at $c/\lambda_{\max}$; use Wien's law only with B_λ.

Grey bodies

A real surface radiates a fraction ε of the blackbody value. It also absorbs radiation from its surroundings, so the net power is:

$$P = \varepsilon \sigma A \left(T^4 - T_{\text{env}}^4\right)$$

- Every temperature must be in kelvin; Celsius values make the answer wrong by orders of magnitude.
- A good absorber is a good emitter: the same ε appears in both terms.

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Bohr scaling

For one electron around a nucleus of charge Ze:

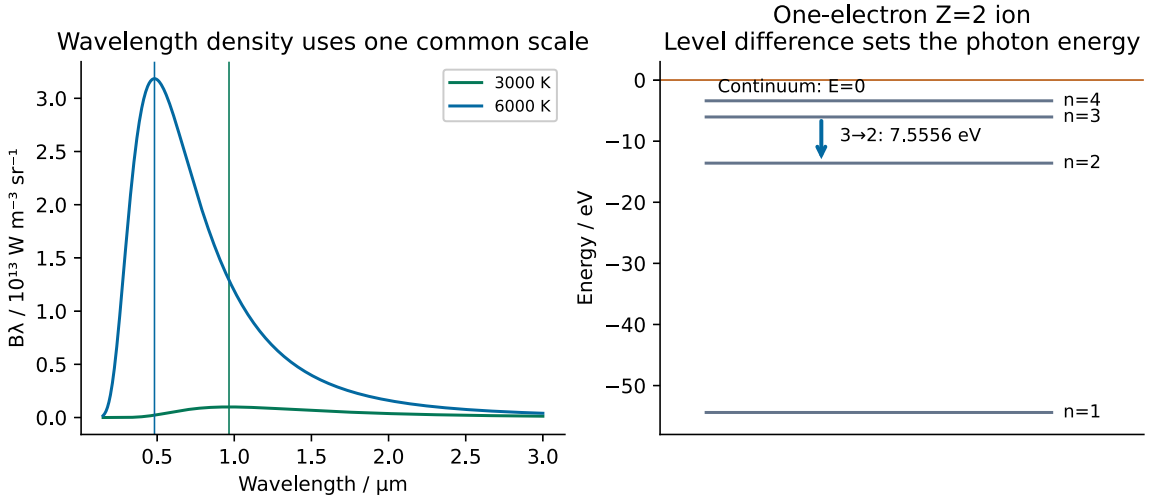
$$r_n = \frac{a_0 n^2}{Z}, \qquad v_n = \frac{Z\alpha c}{n}, \qquad E_n = -\frac{13.6\,Z^2}{n^2}\ \text{eV}$$

- Radius and energy carry different powers of Z: He^+ is half the size of hydrogen and four times as tightly bound.
- The Bohr model gets the energies right but not the angular momentum: the real 1s state has $l = 0$, not $L = \hbar$.

Lines, limits and ionisation

$$E_\gamma = 13.6\,Z^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)\ \text{eV}, \qquad \lambda = \frac{1240\ \text{eV nm}}{E_\gamma}$$

- The series limit is the photon with $n_i \to \infty$: for Balmer in hydrogen, $13.6/4 = 3.40$ eV, 365 nm.
- Ionising from level n needs $13.6\,Z^2/n^2$, which is not the same as any transition energy.



Doubling the temperature halves the peak wavelength; for a $Z = 2$ ion every level is $Z^2 = 4$ times the hydrogen value

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Worked example: a hot filament

A tungsten filament at 3000 K has emissivity 0.5 and area 2.0×10^{-4} m^2, in cold surroundings.

- Peak: $\lambda_{\max} = 2.898 \times 10^{-3}/3000 = 966$ nm, in the infrared.
- Power: $0.5 \times 5.67 \times 10^{-8} \times 2.0 \times 10^{-4} \times 3000^4 = 459$ W.
- A photon at the peak carries $1240/966 = 1.28$ eV.

Worked example: the helium ion

- He$^+$ $3 \to 2$: $E_\gamma = 54.4\left(\frac{1}{4} - \frac{1}{9}\right) = 54.4 \times \frac{5}{36} = 7.56$ eV, so $\lambda = 1240/7.56 = 164$ nm.
- Ionising He$^+$ from $n = 3$ needs $54.4/9 = 6.04$ eV.
- Its $n = 2$ orbit has radius $a_0 \times 4/2 = 2a_0$, half of hydrogen's $4a_0$.

Common errors

- Using Celsius in T^4.
- Forgetting the surroundings in the net radiated power.
- Using Wien's wavelength peak to find the frequency peak.
- Giving radius and energy the same power of Z.
- Confusing the ionisation energy from level n with a transition energy.

Key terms

blackbody: An ideal surface that absorbs all radiation and emits a spectrum set only by its temperature.

emissivity: The ratio of a surface's thermal radiation to that of a blackbody at the same temperature.

spectral radiance: Radiated power per unit area, per unit solid angle and per unit wavelength (or frequency).

series limit: The shortest-wavelength line of a series, from an initial level at $n \to \infty$.

Check yourself

- A star's spectrum peaks at 290 nm. What is its surface temperature?
- By what factor does the radiated power rise when the temperature of a blackbody doubles?
- What is the ground-state energy of Li^{2+} ($Z = 3$)?

Answers: about 10,000 K. 16. $-13.6 \times 9 = -122$ eV.

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Simultaneity, length measurement and velocity transformation

GRE Subject Test · GRE Physics

Lorentz transformations, simultaneity and the interval

In June 1905 Albert Einstein began "On the Electrodynamics of Moving Bodies" with what looked like a trivial question: what does it mean to say two events happen at the same time? He answered with clocks synchronised by light signals. With that definition, observers moving relative to each other disagree about which distant events are simultaneous. Lengths and time intervals then also depend on the frame. Only a few combinations, such as the spacetime interval, are the same for everyone.

GRE Physics relativity questions ask you to:

- Apply the **Lorentz transformation** 洛伦兹变换 to the separation between two events.
- Show that **simultaneity** 同时性 depends on the frame.
- Measure a moving length correctly.
- Classify separations with the **spacetime interval** 时空间隔.
- Add velocities relativistically.

Transforming separations

Let frame S' move at speed v along $+x$ relative to S, with $\gamma = 1/\sqrt{1 - v^2/c^2}$:

$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right), \qquad \Delta t' = \gamma\left(\Delta t - \frac{v\,\Delta x}{c^2}\right)$$

- Always transform Δx and Δt together; the time result depends on the distance too.
- Two events simultaneous in S ($\Delta t = 0$) but Δx apart have $\Delta t' = -\gamma v \Delta x / c^2$ in S': the event further along $+x$ happens first.

Measuring a moving rod

A length is the distance between both ends read at the same time in your own frame.

- If that condition holds, $L = L_0/\gamma$, where L_0 is the **proper length** 固有长度, measured in the rod's rest frame.
- At $0.6c$, $\gamma = 1.25$, so a 2.0 m rod is 1.6 m long in the lab.
- Reading the two ends at different times gives a number that is not a length.

The invariant interval

$$s^2 = c^2\Delta t^2 - \Delta x^2$$

This is the same in every inertial frame.

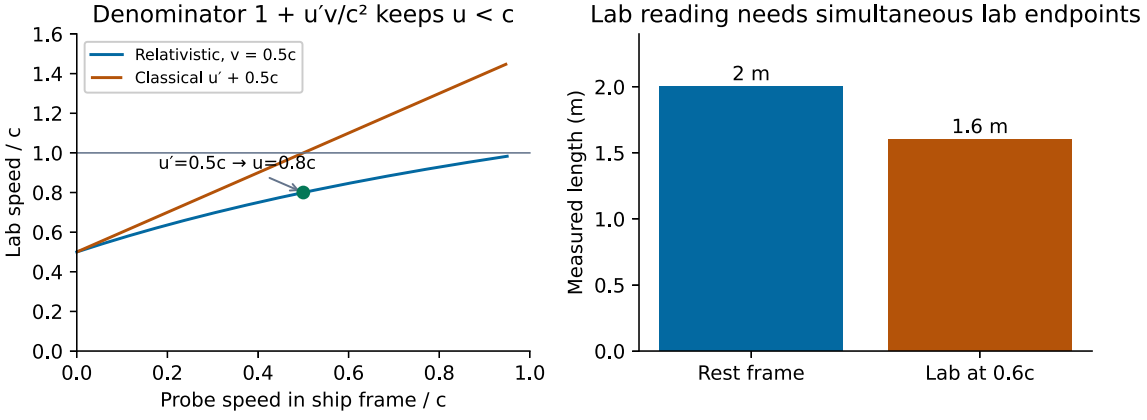
- $s^2 > 0$, timelike: one event can cause the other, the order is the same in every frame, and the **proper time** 固有时 between them is s/c.
- $s^2 < 0$, **spacelike** 类空: no signal can connect them, and different frames can disagree about their order.
- $s^2 = 0$, lightlike: only a light signal connects them.

Velocity addition

An object moves at u' in S'. Its speed in S is:

$$u = \frac{u' + v}{1 + u'v/c^2}$$

- A probe launched at $0.5c$ from a ship moving at $0.5c$ moves at $1.0c/1.25 = 0.8c$.
- With $u' = c$, $u = c$: light has the same speed in every frame.
- At low speeds the denominator is close to 1 and the result is $u' + v$.



Velocity addition keeps u below c (0.5c with 0.5c gives 0.8c); length contracts by 1/γ

Worked example: two lightning strikes

Two strikes hit the ground at the same time, 900 m apart. A train moves at $0.6c$ toward the far strike.

- $c\Delta t' = -\gamma(v/c)\Delta x = -1.25 \times 0.6 \times 900 = -675$ m, so $\Delta t' = -675/(3.0 \times 10^8) = -2.25$ μs.
- The passenger sees the forward strike 2.25 μs before the rear one. Both observers are right in their own frames.
- $s^2 = 0 - 900^2 < 0$: spacelike, so no frame can make one strike cause the other.

Worked example: classifying a pair

Two events have $\Delta x = 900$ m and $c\Delta t = 600$ m.

- $s^2 = 600^2 - 900^2 < 0$: spacelike.
- A frame with $v = c^2\Delta t/\Delta x = \frac{2}{3}c$ sees them simultaneous, $\sqrt{900^2 - 600^2} = 671$ m apart.
- Change to $\Delta x = 600$ m, $c\Delta t = 1000$ m: timelike, with $s = 800$ m and proper time $800/(3.0 \times 10^8) = 2.67$ μs.

Common errors

- Transforming Δt without the $v\Delta x/c^2$ term.
- Measuring a moving rod's ends at different times.
- Contracting the proper length instead of the moving one, or contracting in the wrong direction.
- Adding velocities as $u' + v$ near c.
- Thinking a spacelike pair can be causally connected.

Key terms

Lorentz transformation: The relativistic rule that converts event coordinates between inertial frames.

simultaneity: Two events happening at the same time in a given frame.

spacetime interval: $s^2 = c^2\Delta t^2 - \Delta x^2$, the same in every inertial frame; its sign classifies a separation.

proper length: The length of an object measured in its own rest frame.

proper time: The time between two events measured by a clock present at both.

spacelike: Too far apart in space, for their time separation, for any signal to connect two events.

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Check yourself

- Two events have $\Delta x = 300$ m and $c\Delta t = 500$ m. Is the pair timelike or spacelike?
- A rod has proper length 5.0 m. How long is it at $0.8c$?
- What is $0.8c$ added to $0.8c$?

Answers: timelike, with $s = 400$ m. 3.0 m. $1.6c/1.64 = 0.976c$.

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Measurement loading, amplifier gain and detector response

GRE Subject Test · GRE Physics

Laboratory instruments, meters and detectors

In 1928 Hans Geiger and his student Walther Müller, in Kiel, built a gas-filled tube that produced a large electrical pulse for each ionising particle passing through it. The Geiger–Müller counter made counting radiation simple, but it has a weakness: after each pulse the tube is "blind" for about 100 microseconds. At high count rates it misses events, and the true rate must be corrected. Every instrument changes, delays or distorts what it measures in some way, and GRE laboratory questions test whether you can correct for it.

GRE Physics laboratory questions ask you to:

- Correct a meter reading for **loading** 负载效应.
- Find the gains of ideal op-amp circuits.
- Use the **cutoff frequency** 截止频率 of an RC filter.
- Report detector efficiency and energy resolution.
- Correct a count rate for **dead time** 死时间.

Meter loading

A voltmeter with resistance R_m across a source with Thévenin resistance R_{th} forms a voltage divider:

$$V_{\text{read}} = V_{\text{th}}\,\frac{R_m}{R_m + R_{\text{th}}}$$

- The reading is always low. A meter with $R_m = 10R_{\text{th}}$ reads 90.9% of the true value.
- A meter's **input impedance** 输入阻抗 should be much larger than the source resistance. A 10× oscilloscope probe raises it, but also divides the signal by 10.
- An ammeter loads a circuit the other way: its resistance must be small.

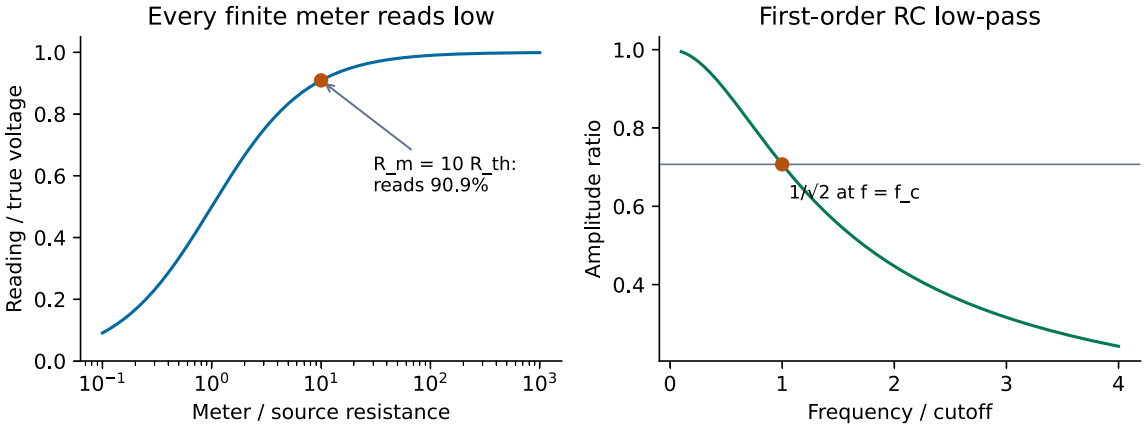
Op-amps and filters

For an ideal op-amp with negative feedback, the inputs draw no current and sit at the same voltage.

- Non-inverting gain: $1 + R_2/R_1$. Inverting gain: $-R_2/R_1$.
- An RC low-pass filter has $f_c = \dfrac{1}{2\pi RC}$.

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{\sqrt{1+(f/f_c)^2}}, \qquad \phi = -\arctan\frac{f}{f_c}$$

- At f_c the amplitude ratio is $1/\sqrt{2} = 0.707$ (half the power), not $\frac{1}{2}$, and the phase is $-45°$.



A finite meter resistance always reads low; a low-pass filter passes 1/√2 at f_c

Detectors

- Efficiency = particles detected ÷ particles reaching the detector.
- Energy resolution = full width at half maximum ÷ energy. A 13.2 keV width at 662 keV is 2.0%.
- These are different properties: a detector can count well and still blur energies.

Dead time

After each recorded event, a non-paralysable detector cannot record another for a time τ. With true rate n, the recorded rate m is:

$$m = \frac{n}{1+n\tau}, \qquad n = \frac{m}{1-m\tau}$$

- The correction matters when $m\tau$ is not small.

Calibration and errors

- A linear calibration fixes an offset and a scale. It does not remove noise or nonlinearity.
- Averaging N readings reduces random error by $\sqrt{N}$, but leaves a systematic bias unchanged.

Worked example: a loaded source

- A 10 V source with 100 kΩ internal resistance, read by a 1.0 MΩ meter: $10 \times 1.0/1.1 = 9.09$ V.
- A 12 V source with 200 kΩ, the same meter: $12 \times 1.0/1.2 = 10.0$ V, a 17% error.
- A non-inverting amplifier with $R_1 = 1.0$ kΩ and $R_2 = 9.0$ kΩ has gain 10.

Worked example: a filter and a counter

$R = 1.6$ kΩ and $C = 100$ nF.

- $f_c = 1/(2\pi \times 1600 \times 1.0 \times 10^{-7}) = 995$ Hz, about 1.0 kHz.
- At 3.0 kHz: ratio $1/\sqrt{1+9} = 0.32$, phase $-\arctan 3 = -72°$.
- A counter with $\tau = 20$ μs records 8000 per second: $n = 8000/(1 - 0.16) = 9524$ per second.

Common errors

- Taking the meter reading as the source voltage.
- Using $\frac{1}{2}$ for the amplitude at the cutoff frequency.
- Mixing up the inverting and non-inverting gains.
- Using $m(1 + m\tau)$ instead of $m/(1 - m\tau)$ for dead time.
- Expecting averaging to remove a systematic error.

Key terms

loading: The change a measuring instrument makes to the circuit it measures.

input impedance: The effective load that a measuring instrument presents to the circuit under test.

cutoff frequency: The frequency at which a filter passes $1/\sqrt{2}$ of the input amplitude.

dead time: The time after one recorded event during which a detector cannot record another.

Check yourself

- A meter with $R_m = 4R_{\text{th}}$ reads a 5.0 V source. What does it show?
- What is the phase shift of an RC low-pass filter at its cutoff frequency?
- A detector with $\tau = 10$ μs has a true rate of 10^4 per second. What rate does it record?

Answers: 4.0 V. $-45°$. $10^4/1.1 = 9.1 \times 10^3$ per second.

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Radiation attenuation, laser gain and cavity modes

GRE Subject Test · GRE Physics

Radiation in matter and laser cavities

On 16 May 1960 Theodore Maiman, at Hughes Research Laboratories in California, fired a bright flash lamp around a small rod of pink ruby with silvered ends. A pulse of deep red light at 694 nm came out of one end, far narrower in colour and direction than anything a lamp could produce: the first laser. Many experts had thought ruby could not work. Maiman showed that the flash lamp could pump enough chromium ions into an upper level to create a **population inversion** 粒子数反转.

GRE Physics questions on radiation and lasers ask you to:

- Find the range of charged particles.
- Use exponential **attenuation** 衰减 and the **half-value layer** 半值层 for photons.
- Explain stimulated emission and the need for population inversion.
- Find the frequencies and spacing of **cavity modes** 腔模.

Charged particles: a definite range

Heavy charged particles such as alpha particles lose energy in many small collisions with electrons, so they slow down steadily and stop.

- With a constant energy loss per distance, range $= E_0/(dE/dx)$. A 5.0 MeV particle losing 2.0 MeV/cm stops after 2.5 cm.
- Real energy loss grows as the particle slows, so most of the energy is deposited near the end of the track.
- Electrons scatter strongly and radiate, so their range is less sharp.

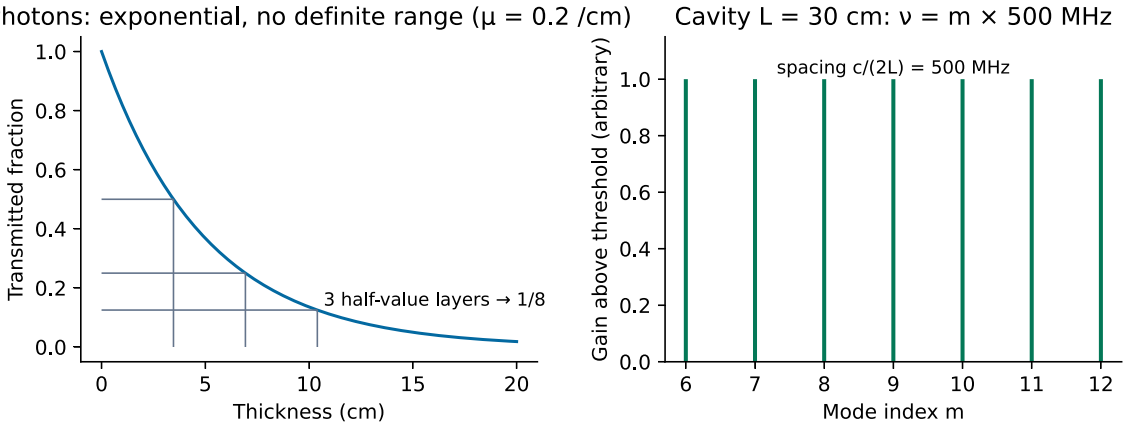
Photons: exponential attenuation

A photon is removed in a single interaction or not at all, so a narrow beam falls exponentially and has no definite range:

$$I = I_0\, e^{-\mu x}, \qquad x_{1/2} = \frac{\ln 2}{\mu}$$

- With $\mu = 0.20\ \text{cm}^{-1}$, the half-value layer is 3.5 cm, and 10 cm transmits $e^{-2} = 0.135$.
- Three half-value layers transmit $\frac{1}{8}$.
- A broad beam transmits more than this, because scattered photons still reach the detector (build-up).

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Photons have no definite range; cavity modes are spaced $c/(2L)$

Stimulated emission and inversion

A photon of the right energy can make an excited atom emit a second photon with the same frequency, direction and phase. This is **stimulated emission** 受激辐射.

- Absorption competes with it, so net gain needs $N_2/g_2 > N_1/g_1$, where g is each level's degeneracy.
- In thermal equilibrium the upper level always has fewer atoms, so heating a two-level system never gives an inversion.
- Lasers pump a third level that decays quickly into a long-lived upper laser level.

Cavity modes

The mirrors of a laser of length L allow only standing waves with $L = m\lambda/2$:

$$\nu_m = \frac{mc}{2L}, \qquad \Delta\nu = \frac{c}{2L}$$

- A 30 cm cavity has modes 500 MHz apart.
- A mode lases only if its gain is larger than its round-trip losses.

Worked example: a shield

A gamma beam meets a shield with $\mu = 0.20\ \text{cm}^{-1}$.

- Half-value layer: $0.693/0.20 = 3.5$ cm.
- After 10 cm: $e^{-2.0} = 0.135$ of the beam remains.
- For a $\frac{1}{16}$ count rate with a detector of efficiency 0.25, the shield need only transmit $\frac{1}{4}$: thickness $\ln 4/\mu = 6.9$ cm.

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Worked example: a laser

- Two levels 2.0 eV apart at 300 K ($k_B T = 0.0259$ eV): $N_2/N_1 = e^{-77} \approx 10^{-34}$. Without pumping, almost no atoms are excited.
- A 25 cm cavity: $\Delta\nu = 3.0 \times 10^8 / 0.50 = 600$ MHz.
- If the gain is above the losses over 1.5 GHz, about $1.5/0.60 \approx 2$ or 3 modes can lase.

Common errors

- Giving photons a definite range, or charged particles an exponential one.
- Using μ in place of $\ln 2/\mu$ for the half-value layer.
- Leaving detector efficiency out of a count rate.
- Thinking heating can produce a population inversion.
- Using c/L for the mode spacing; the round trip is $2L$.

Key terms

attenuation: The reduction in intensity of a beam as it passes through matter.

half-value layer: The thickness of material that halves a narrow beam, $\ln 2/\mu$.

population inversion: A state with more atoms per sublevel in the upper laser level than the lower.

stimulated emission: Emission caused by an incoming photon, giving a copy of that photon.

cavity modes: The standing-wave frequencies a laser's mirrors allow.

Check yourself

- A beam passes through four half-value layers. What fraction remains?
- What is the mode spacing of a 1.5 m laser cavity?
- Why can a two-level system in thermal equilibrium not lase?

Answers: $\frac{1}{16}$. 100 MHz. Its upper level always has fewer atoms than its lower one.

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Bragg diffraction and electron-gas heat capacity

GRE Subject Test · GRE Physics

Crystal diffraction and free-electron metals

In 1912 Max von Laue's group in Munich sent X-rays through a copper sulfate crystal and found a pattern of spots on a photographic plate. Later that year Lawrence Bragg, a 22-year-old student at Cambridge, saw a simpler way to read such patterns: treat the crystal as stacks of parallel planes of atoms, each reflecting a little of the beam. In 1913 he and his father William used this idea to work out the structure of sodium chloride. They shared the 1915 Nobel Prize; Lawrence, at 25, is still the youngest physics laureate.

GRE Physics solid-state questions ask you to:

- Use **Bragg's law** 布拉格定律 with the correct angle.
- Find plane spacings in a cubic crystal from its **lattice constant** 晶格常数.
- Use the free-electron model to find the **Fermi energy** 费米能.
- Separate the electron and lattice parts of a metal's heat capacity.

Bragg's law

Waves reflected from neighbouring planes a distance d apart add in phase when the extra path is a whole number of wavelengths:

$$2d\sin\theta = n\lambda$$

- θ is the glancing angle measured from the plane, not from the normal. A detector reading 2θ shows the angle between the incident and scattered beams.
- Because $\sin\theta \le 1$, only orders with $n \le 2d/\lambda$ appear.

Planes in cubic crystals

A family of planes is named by its Miller indices (hkl). In a cubic crystal with lattice constant a:

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

- (100) planes are a apart; (110) planes are $a/\sqrt{2}$ apart; (111) planes are $a/\sqrt{3}$ apart.
- Some reflections are missing because atoms inside the cell cancel them. Body-centred cubic: $h+k+l$ odd is absent. Face-centred cubic: only all-even or all-odd hkl appear.

Free electrons

Treat the conduction electrons as a gas of fermions with number density n in a box:

$$k_F = \left(3\pi^2 n\right)^{1/3}, \qquad E_F = \frac{\hbar^2 k_F^2}{2m}$$

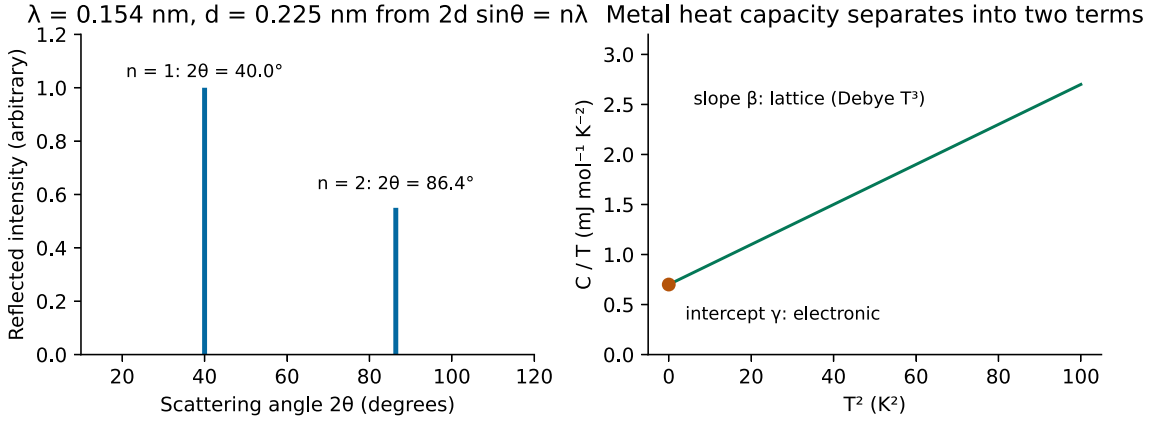
- For metals E_F is several eV, so the Fermi temperature $T_F = E_F/k_B$ is tens of thousands of kelvin.
- Only electrons within about $k_B T$ of E_F can gain energy, a fraction of roughly T/T_F.

Heat capacity of a metal

At low temperature a metal's heat capacity has an electron term and a lattice term:

$$C = \gamma T + \beta T^3$$

- Plot C/T against T^2: a straight line with intercept γ and slope β.
- The electron term wins at the lowest temperatures; at room temperature the lattice dominates, near the Dulong–Petit value $3R$.



Bragg's law $2d \sin\theta = n\lambda$; $C/T = \gamma + \beta T^2$ separates electrons from lattice

Worked example: reading a diffraction pattern

Copper K-alpha X-rays ($\lambda = 0.154$ nm) give a first-order peak at $\theta = 20.0°$.

- $d = 0.154/(2\sin 20.0°) = 0.154/0.684 = 0.225$ nm.
- Second order: $\sin\theta = 0.154/0.225 = 0.684$, so $\theta = 43.2°$ and $2\theta = 86.4°$. There is no third order, since $3\lambda/2d > 1$.
- Rock salt ($a = 0.564$ nm) has $d_{200} = 0.282$ nm, giving $\sin\theta = 0.273$ and $\theta = 15.8°$.

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Worked example: copper's electrons

Copper has $n = 8.5 \times 10^{28}$ m^{-3}.

- $k_F = (3\pi^2 \times 8.5 \times 10^{28})^{1/3} = 1.36 \times 10^{10}$ m^{-1}.
- $E_F = (1.055 \times 10^{-34})^2 (1.36 \times 10^{10})^2 / (2 \times 9.11 \times 10^{-31}) = 1.13 \times 10^{-18}$ J $= 7.0$ eV, so $T_F = 8.1 \times 10^4$ K.
- If $C/T = 0.7 + 0.05T^2$ (mJ mol^{-1} K^{-2}), the two terms are equal where $0.05T^2 = 0.7$, at $T = 3.7$ K.

Common errors

- Measuring θ from the normal, or using the detector angle 2θ as θ.
- Allowing orders with $n\lambda > 2d$.
- Forgetting the systematic absences of BCC and FCC lattices.
- Expecting every electron to share the thermal energy; only a fraction T/T_F does.
- Fitting C against T instead of C/T against T^2.

Key terms

Bragg's law: The condition $2d\sin\theta = n\lambda$ for strong reflection from crystal planes, with θ measured from the planes.

lattice constant: The side length of a crystal's repeating cubic cell.

Fermi energy: The highest occupied electron energy at absolute zero in the free-electron model.

Check yourself

- A cubic crystal has $a = 0.30$ nm. What is d_{111}?
- Does a body-centred cubic crystal show a (100) reflection?
- A metal's heat capacity is measured at 1 K. Which term usually dominates?

Answers: $0.30/\sqrt{3} = 0.17$ nm. No: $h + k + l = 1$ is odd. The electron term, γT.

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Reaction energies, activity and decay balances

GRE Subject Test · GRE Physics

Nuclear energy release, activity and decay chains

Just before Christmas 1938, Otto Hahn and Fritz Strassmann in Berlin found barium among the products of uranium bombarded with neutrons. Barium is about half as heavy as uranium. Lise Meitner, who had fled to Sweden, and her nephew Otto Frisch worked out over a walk in the snow that the uranium nucleus had split. From the binding energies they estimated about 200 MeV per split, an enormous energy for one atom. Frisch called the process "fission" 裂变. Nuclear questions start from the same idea: a small **mass defect** 质量亏损 is a large energy.

GRE Physics nuclear questions ask you to:

- Find the **Q value** Q 值 of a reaction from atomic masses.
- Balance charge and nucleon number.
- Use **activity** 活度, the decay constant and half-life.
- Follow decay chains to **secular equilibrium** 长期平衡.
- Read the binding-energy curve.

Q values

$$Q = (m_{\text{initial}} - m_{\text{final}})\,c^2, \qquad 1\ \text{u} \to 931.5\ \text{MeV}$$

- $Q > 0$: kinetic energy is released. $Q < 0$: energy must be supplied.
- Tables give atomic masses, which include electrons. They cancel for alpha and beta-minus decay. For beta-plus decay subtract $2m_e c^2 = 1.022$ MeV.

Balance first

Before any arithmetic, check that charge (the lower number Z) and nucleon number (the upper number A) are the same on both sides.

- ${}^{14}_{7}\text{N} + {}^{4}_{2}\text{He} \to {}^{17}_{8}\text{O} + {}^{1}_{1}\text{p}$: charge $7 + 2 = 8 + 1$, nucleons $14 + 4 = 17 + 1$.
- Alpha decay lowers Z by 2 and A by 4; beta-minus raises Z by 1 and leaves A unchanged.

Activity

$$A = \lambda N, \qquad \lambda = \frac{\ln 2}{T_{1/2}}, \qquad A(t) = A_0 e^{-\lambda t}$$

- Activity is measured in becquerel: 1 Bq = 1 decay per second.

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- Two samples with equal activities can hold very different numbers of atoms, if their half-lives differ.

Decay chains

For a chain $A \to B \to C$:

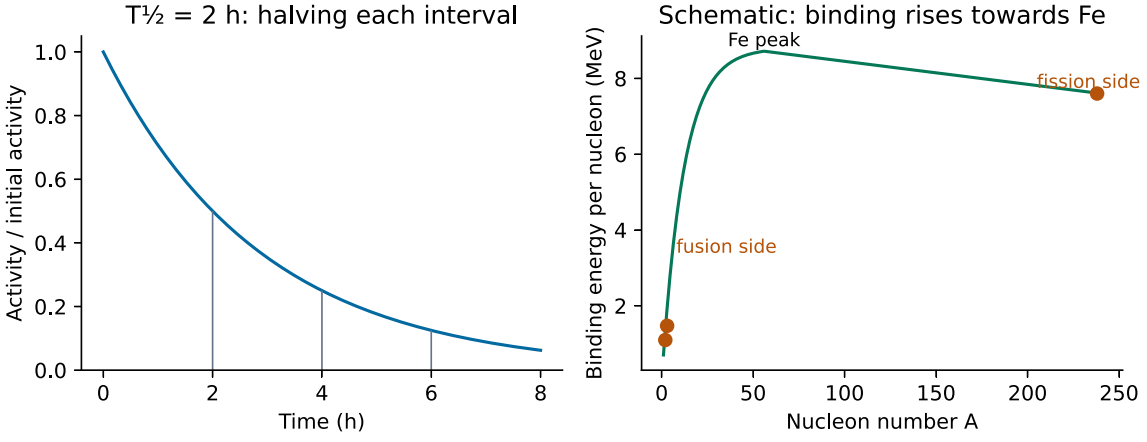
$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B$$

- If the parent lives much longer than the daughter, the daughter's activity rises to equal the parent's: secular equilibrium.
- Radium-226 (1600 years) and radon-222 (3.8 days) reach it within about a month.

The binding curve

Binding energy per nucleon rises steeply for light nuclei, peaks near iron at about 8.8 MeV, then falls slowly.

- Fusing light nuclei or splitting heavy ones both move toward the peak and release energy.
- Fission of $A \approx 240$: about $(8.5 - 7.6) \times 240 \approx 200$ MeV.



Each half-life halves the activity; energy is released by moving towards the iron peak, by fusion from the left or fission from the right

Worked example: Q values

- $^{238}\mathrm{U} \to {}^{234}\mathrm{Th} + \alpha$: $(238.0508 - 234.0436 - 4.0026) \times 931.5 = 0.0046 \times 931.5 = 4.3$ MeV.
- Momentum is shared equally, so the light alpha takes about $\frac{234}{238}$ of it, 4.2 MeV.
- $\mathrm{D} + \mathrm{T} \to {}^{4}\mathrm{He} + \mathrm{n}$: $(2.0141 + 3.0160 - 4.0026 - 1.0087) \times 931.5 = 0.0188 \times 931.5 = 17.5$ MeV.

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Worked example: activity

A sample has 6.0×10^{12} atoms and a half-life of 2.0 h.

- $\lambda = 0.693/7200 = 9.6 \times 10^{-5}$ s^{-1}.
- $A = 9.6 \times 10^{-5} \times 6.0 \times 10^{12} = 5.8 \times 10^{8}$ Bq.
- After 6.0 h, three half-lives, the activity is $\tfrac{1}{8}$ of this, 7.2×10^{7} Bq.

Common errors

- Computing Q before checking the balance.
- Leaving out the two electron masses in beta-plus decay.
- Using the half-life in place of λ in $A = \lambda N$.
- Assuming equal activities mean equal numbers of atoms.
- Thinking only fission releases energy, or that fusing iron does.

Key terms

fission: The splitting of a heavy nucleus into two lighter ones.

mass defect: The difference between the mass of the separate parts and the mass of the bound system.

Q value: The rest energy turned into kinetic energy in a reaction, $(m_{\text{initial}} - m_{\text{final}})c^2$.

activity: The decay rate λN of a radioactive sample, in becquerel.

secular equilibrium: The state in which a short-lived daughter's activity equals that of its long-lived parent.

Check yourself

- What particle is missing in ${}^{9}_{4}\mathrm{Be} + {}^{4}_{2}\mathrm{He} \to {}^{12}_{6}\mathrm{C} + $?
- A sample's activity falls to $\tfrac{1}{16}$ in 20 days. What is its half-life?
- Why does fusing two iron nuclei not release energy?

Answers: a neutron, ${}^{1}_{0}\mathrm{n}$. 5 days. Iron is at the peak of the binding curve.

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Coordinate operators, separated modes and stable updates

GRE Subject Test · GRE Physics

Coordinate operators, separation of variables and stable steps

In 1928 Richard Courant, Kurt Friedrichs and Hans Lewy in Göttingen asked when a step-by-step computer-style calculation of a wave actually follows the true wave. Their answer, now called the CFL condition, was surprising: if the time step is too large for the grid, the numbers grow without limit, however accurate each single step looks. Mathematical methods questions test the same kind of care: use the operator that belongs to your coordinates, use the boundary conditions, and choose a stable step.

GRE Physics mathematical-methods questions ask you to:

- Use the **Laplacian** 拉普拉斯算子 and divergence in spherical and cylindrical coordinates.
- Solve wave and heat equations by **separation of variables** 分离变量法.
- Find the **stability** 稳定性 limit of the explicit Euler method.
- Check that a numerical answer is accurate, not only stable.

Radial operators

For functions of r alone, in spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = f'' + \frac{2}{r}f', \qquad \nabla\cdot(A_r\hat{r}) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 A_r\right)$$

- $1/r$ has zero Laplacian for $r>0$: the potential of a point charge.
- $\hat{r}/r^2$ has zero divergence for $r>0$: the field of a point charge.
- r^2 has Laplacian $2+4=6$.
- In cylindrical coordinates the radial Laplacian is $\frac{1}{s}\frac{d}{ds}\left(s\frac{df}{ds}\right)$, so $\ln s$ is the harmonic function.

Separation of variables

For a string fixed at both ends, $u_{tt} = c^2 u_{xx}$, try $u = X(x)T(t)$. Then $X''/X = T''/(c^2T) = -k^2$, a constant.

- $X(0) = X(L) = 0$ forces $X = \sin(n\pi x/L)$, so the frequencies are $f_n = nc/2L$.
- The general solution is a sum of these modes, with amplitudes set by the initial shape.
- For the heat equation $u_t = Du_{xx}$, each mode decays as $e^{-D(n\pi/L)^2 t}$: high modes die out fastest.

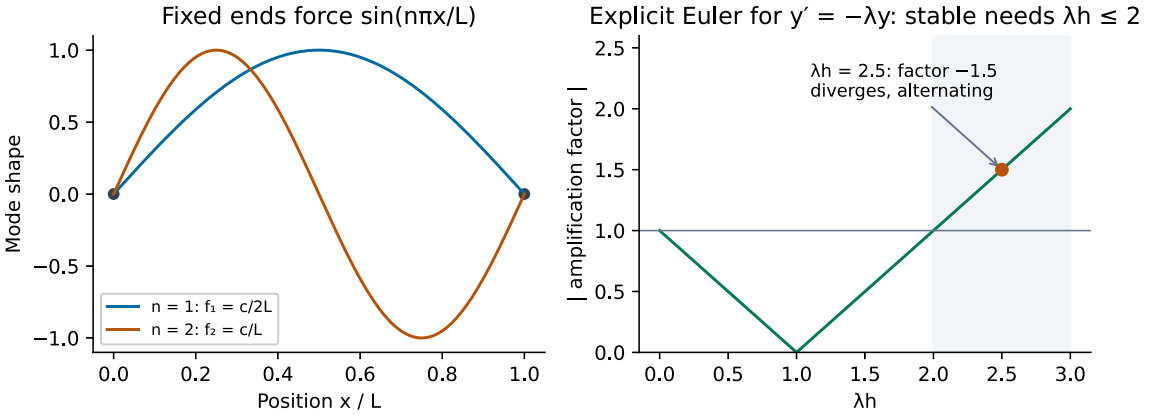
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Explicit Euler stability

For $y' = -\lambda y$ with step h:

$$y_{n+1} = (1 - \lambda h)\, y_n$$

- The method is stable when $|1 - \lambda h| \le 1$, that is $0 < h \le 2/\lambda$.
- For $1/\lambda < h < 2/\lambda$ the result oscillates in sign but still decays; above $2/\lambda$ it grows.



Fixed ends force sin(nπx/L); explicit Euler for $y' = -\lambda y$ is stable only for $\lambda h \le 2$

Stable is not accurate

- Explicit Euler has an error roughly proportional to h. Halve the step and the error should roughly halve.
- Check that λh is dimensionless, compare two step sizes, and test a case with a known answer.

Worked example: operators and modes

- For $f = 1/r$: $f'' = 2/r^3$ and $\frac{2}{r}f' = -2/r^3$, so $\nabla^2 f = 0$. Using f'' alone gives the wrong $2/r^3$.
- A string with $L = 0.50$ m and $c = 100$ m/s: $f_1 = 100/(2 \times 0.50) = 100$ Hz, and $f_3 = 300$ Hz.
- $\lambda = 10\ \text{s}^{-1}$ with $h = 0.25$ s: the factor is $1 - 2.5 = -1.5$, so the solution grows with alternating sign.

Worked example: refining a step

Solve $y' = -4y$, $y(0) = 8$ to $t = 1$. The exact answer is $8e^{-4} = 0.147$.

- $h = 0.2$: factor 0.2, five steps, $8 \times 0.2^5 = 0.0026$.
- $h = 0.1$: $8 \times 0.6^{10} = 0.048$. $h = 0.05$: $8 \times 0.8^{20} = 0.092$.
- All three are stable ($h < 0.5$), but only small steps approach the right answer.

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Common errors

- Writing $\nabla^2 f = f''$ in spherical or cylindrical coordinates.
- Summing modes that do not satisfy the boundary conditions.
- Treating $h < 2/\lambda$ as a guarantee of accuracy.
- Mixing units, so that λh is not dimensionless.
- Forgetting that high modes of the heat equation decay fastest.

Key terms

Laplacian: ∇^2, the sum of second derivatives, written in the form that belongs to the coordinate system.

separation of variables: Solving a partial differential equation by writing the unknown as a product of one-variable factors.

stability: A numerical method's property of not amplifying errors from step to step.

Check yourself

- What is the Laplacian of r^3 in spherical coordinates?
- What is the largest stable explicit Euler step for $y' = -50y$?
- A string fixed at both ends has $f_1 = 220$ Hz. What is f_2?

Answers: $6r + 6r = 12r$. $h = 0.04$. 440 Hz.

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