

Finite-well bound states and hydrogenic quantum numbers

GRE · GRE Subject Test · GRE Physics

A bound wavefunction can extend outside a finite well even though the particle's energy is below the external potential.

Prerequisites: 5, 24, 37.

- Match decaying finite-well states and derive parity-dependent quantisation conditions
- Interpret dimensionless bound-state roots and distinguish confinement from scattering
- Use hydrogenic quantum numbers, angular momentum and radial probability measures

Match bound-state asymptotes

Declare a symmetric well $V=-V_0$ for $|x|<a$ and $V=0$ outside, with $V_0>0$ and the same mass throughout. A bound energy lies between $-V_0$ and zero. Inside, $k=\sqrt{2m(E+V_0)}/\hbar$ gives oscillatory solutions; outside, $\kappa=\sqrt{-2mE}/\hbar$ gives decaying tails. Reject growing exponentials to obtain a normalisable state. Reflection symmetry permits even interior $\cos(kx)$ or odd $\sin(kx)$ states. At finite boundaries without delta interactions, both ψ and ψ' are continuous. Even matching gives $k \tan(ka)=\kappa$; odd matching gives $-k \cot(ka)=\kappa$. A finite well does not require ψ to vanish at its edges as an infinite wall does.

Solve a dimensionless root

Introduce $z=ka$ and $\rho=a\sqrt{2mV_0}/\hbar$. Then $\kappa a=\sqrt{\rho^2-z^2}$ and $0<z<\rho$. Solve $z \tan z=\sqrt{\rho^2-z^2}$ for an even state, or $-z \cot z=\sqrt{\rho^2-z^2}$ for an odd one, on intervals with the appropriate sign and away from tangent poles. Energy follows as $E/V_0=z^2/\rho^2-1$. For $\rho=1$, the ground root lies in $0<z<1$ and the even equation is equivalent there to $z=\cos z$. Bisection or a converged root finder gives $z\approx 0.739085$ and $E/V_0\approx -0.453753$. Since $\rho<\pi/2$, no odd bound-state branch fits. In one dimension an attractive square well has an even bound ground state however shallow it is; an $E=0$ threshold tail is not square integrable.

Normalise the interior and tails

For an even bound state write $\psi=A \cos(kx)$ inside and $\psi=A \cos(ka)\exp[-\kappa(|x|-a)]$ outside. Continuity sets the relative tail amplitude; integrate $|\psi|^2$ over both the interior and tails to determine A . Probability outside the well is nonzero and depends on its depth, width and the selected state. Bound states have discrete energies and

normalisable decaying asymptotes. A scattering state with $E>0$ has propagating external waves and is normalised or interpreted using a continuum/flux convention. A decaying tail is not evidence that the state violates energy conservation, and raw tail amplitude is not the outside probability until it has been integrated and normalised.

Choose the hydrogenic probability measure

For a one-electron Coulomb ion with nuclear charge Ze , use the nonrelativistic central-potential model and a heavy nucleus approximation unless reduced mass is specified. $E_n\approx -13.6 Z^2/n^2$ eV; allowed orbital numbers are $n\geq 1$, $l=0,\dots,n-1$ and $m_l=-l,\dots,l$. L^2 has eigenvalue $l(l+1)\hbar^2$ and $L_z=m_l\hbar$; the ground $1s$ orbital has $l=0$, despite the historical Bohr circular-orbit rule. Spatial degeneracy at fixed n is $\sum_l (2l+1)=n^2$ before spin and fine-structure corrections. If $\psi=R_{nl}(r)Y_{lm}$ with angular part normalised, radial probability in dr is $r^2|R|^2dr$, not just $|R|^2dr$. For hydrogen $1s$, the radial density is $4r^2e^{-(2r/a_0)}/a_0^3$: it peaks at $r=a_0$ and has mean $3a_0/2$, while the three-dimensional point density is largest at the origin. Distinguish these measures before locating a most probable radius.

Worked method

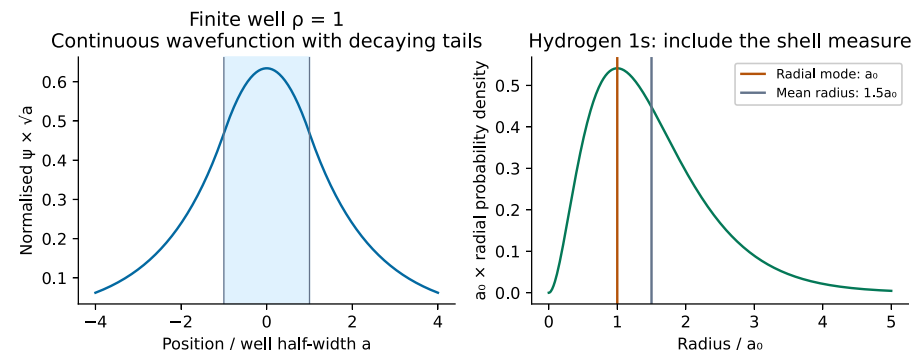
A finite well has $V=-V_0$ for $|x|<a$ and zero outside. A bound energy lies strictly between $-V_0$ and zero. Define

$$k = \sqrt{2m(E + V_0)}/\hbar, \quad \kappa = \sqrt{-2mE}/\hbar.$$

Even parity gives an interior cosine and decaying tails. Matching value and derivative at a gives

$$k \tan(ka) = \kappa.$$

Odd parity instead gives $-k \cot(ka) = \kappa$. Solve only within valid parity intervals; a root at a tangent pole is not a bound state.



Finite-well bound states and hydrogenic quantum numbers: original GRE teaching diagram.

Check conditions and vocabulary

Use the declared potential zero, match ψ' as well as ψ , and include the radial shell measure. A bound tail and a continuum travelling wave have different energy regimes.

bound state: Normalisable stationary state with confined probability and discrete energy in the specified model.

radial probability density : Probability per radial distance, including the spherical-shell measure after angular integration.