

Weak perturbations and degenerate subspaces

GRE · GRE Subject Test · GRE Physics

A zero first-order energy shift can coexist with a changed state and a nonzero higher-order shift.

Prerequisites: 5, 24.

- Calculate nondegenerate first-order energy shifts as normalised expectation values
- Use symmetry and matrix elements to assess weak-coupling limits
- Diagonalise the perturbation within an exactly degenerate subspace

Weight the perturbation by probability

Let $H=H_0+\lambda W$, where λ is a small dimensionless parameter and the eigenstates of H_0 are known and normalised. For a nondegenerate level n , the first-order energy correction is $\lambda\langle n|W|n\rangle$. In position space this is $\lambda\int\psi_n^*(x)W(x)\psi_n(x)dx$ for a multiplicative potential, with integration over the allowed domain. A potential value at one point is not an expectation value; the state's probability density weights the whole domain. For an infinite well $0<x<L$, $\psi_n=\sqrt{2/L}\sin(n\pi x/L)$. Reflection about its centre makes $\langle x\rangle=L/2$, so a weak added potential γx has shift $\gamma L/2$ for every nondegenerate well level at first order. γ has units of energy per length.

Use parity carefully

For a perturbation odd about the centre, such as $\gamma(x-L/2)$, the probability density of an unperturbed well eigenstate is even, so its diagonal expectation vanishes. A centred quadratic perturbation $\eta(x-L/2)^2$ instead gives shift $\eta L^2[1/12-1/(2\pi^2n^2)]$, which is positive for $\eta>0$ and depends on n . Parity makes the first-order integral zero only for the specified state and operator symmetry. Off-diagonal matrix elements can still change the wavefunction. Do not turn a symmetry cancellation into a claim that every energy correction vanishes or that the original state remains exact.

Compare coupling with level gaps

For a nondegenerate state, the leading admixture of another unperturbed state m is proportional to $\lambda W_{mn}/(E_n^0-E_m^0)$. The useful smallness condition therefore compares coupling matrix elements with the relevant level separations, not just with an arbitrary absolute energy zero. To second order, the energy correction contains $\Sigma_{(m\neq n)}|\lambda W_{mn}|^2/(E_n^0-E_m^0)$. For the lowest nondegenerate state, all these denominators are negative, so the second-order correction is nonpositive in this model. Near a degeneracy, a small denominator defeats the nondegenerate expansion; use a coupled subspace instead of dividing by zero.

Resolve a degenerate subspace

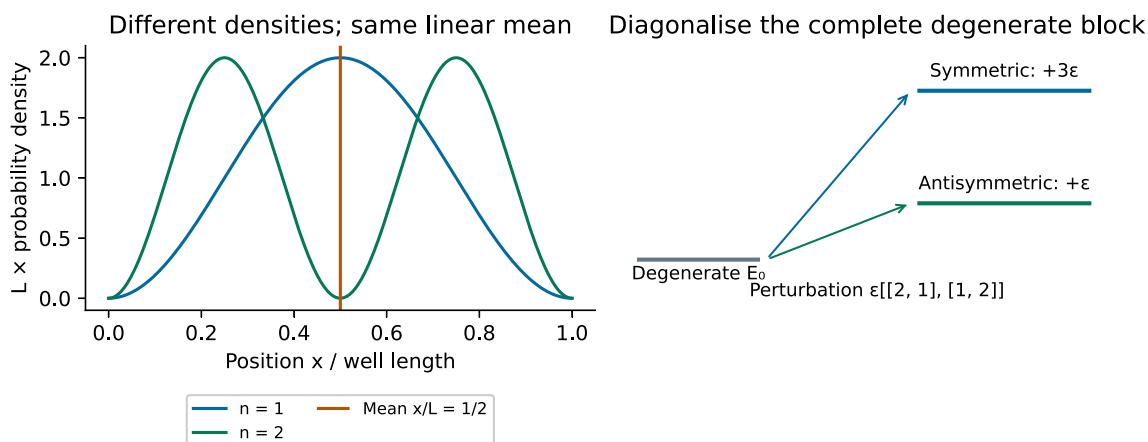
If a level of H_0 is exactly degenerate, choose an orthonormal basis within that subspace and form the Hermitian matrix of the perturbation there. Its eigenvalues are the first-order energy shifts, and its eigenvectors specify the combinations that diagonalise the leading splitting. For a two-state subspace with perturbation $\epsilon[[2,1],[1,2]]$, the normalised symmetric and antisymmetric combinations have shifts 3ϵ and ϵ . Reading only the two diagonal entries would incorrectly predict two shifts of 2ϵ . A common scalar multiple of the identity shifts both states equally and does not split their degeneracy; off-subspace couplings can matter at higher order.

Worked method

A degenerate subspace requires matrix diagonalisation before first-order shifts. Suppose the perturbation restricted to two states is $\epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$$\det(W - wI) = (2\epsilon - w)^2 - \epsilon^2 = 0.$$

Shifts are epsilon and 3 epsilon. Normalised states are $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$. Reading only the diagonal entries would miss the splitting. Couplings to other levels must remain weak relative to their energy gaps.



Weak perturbations and degenerate subspaces: original GRE teaching diagram.

Check conditions and vocabulary

Use normalised state weights, keep perturbation units, and diagonalise a degenerate block. First-order cancellation and exact invariance are different claims.

first-order energy shift: Leading weak-perturbation correction given by the unperturbed state's expectation value.

degenerate perturbation theory : Method that first diagonalises the perturbation within an unperturbed degenerate subspace.