

RC and RL transients, impedance and resonance

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RC and RL transients, impedance and resonance

A capacitor voltage and an inductor current cannot change instantly under finite ordinary circuit drives, but their steady-state behaviour is different.

Prerequisites: 14.

- Solve RC charging and discharging with stated initial conditions
- Solve RL current response and account for stored energy
- Use complex impedance, phase and resonance in sinusoidal circuits

Solve the capacitor response

For a resistor R in series with a capacitor C and a DC source V_s , Kirchhoff's voltage law is $R \, dq/dt + q/C = V_s$. Define $V_C = q/C$ and $\tau = RC$. After a switch to a constant source, $V_C(t) = V_s + (V_C(0) - V_s)\exp(-t/\tau)$. An initially uncharged capacitor therefore has $V_C = V_s(1 - \exp(-t/\tau))$ and charging current $I = (V_s/R)\exp(-t/\tau)$. With the source removed and the resistor connected across it, $V_C = V_C(0)\exp(-t/\tau)$. Capacitor voltage is continuous across the switch if there is no impulsive current; the resistor current may change instantly.

Track inductor current and energy

In a series RL circuit, Kirchhoff's law gives $L \, dI/dt + RI = V_s$ and $\tau = L/R$. The current after a constant drive is $I(t) = V_s/R + (I(0) - V_s/R)\exp(-t/\tau)$. Inductor current is continuous for a finite voltage; its voltage may change abruptly as switching changes dI/dt . At late times an ideal inductor in a DC circuit acts as a zero-voltage connection, while an ideal capacitor has zero DC current. Stored energies are $LI^2/2$ and $CV_C^2/2$. During decay into a resistor, the initially stored energy becomes resistor heat rather than disappearing when the source is disconnected.

Use the phasor convention

Use the declared phasor convention $\exp(i\omega t)$. A resistor has impedance R , an inductor $i\omega L$ and a capacitor $1/(i\omega C) = -i/(\omega C)$. A series RLC circuit therefore has $Z = R + iX$ with $X = \omega L - 1/(\omega C)$. Divide the source voltage phasor by Z to obtain current. Its magnitude is $V_{\text{rms}}/\sqrt{R^2 + X^2}$ when rms quantities are used. The impedance phase φ satisfies $\tan\varphi = X/R$; current lags voltage for $X > 0$ and leads it for $X < 0$. Adding the scalar magnitudes R , ωL and $1/(\omega C)$ loses the vector phase information.

Separate resonance from decay

Series resonance occurs at $\omega_0 = 1/\sqrt{LC}$ when $X=0$. Current is maximal for a fixed voltage in this ideal series model, and the voltage/current phase difference is zero. Mean real power is $V_{\text{rms}} I_{\text{rms}} \cos\varphi = I_{\text{rms}}^2 R$; ideal L and C exchange energy but dissipate no average power. This differs from transient natural frequency: a damped series circuit can oscillate at $\sqrt{1/(LC) - (R/(2L))^2}$ when underdamped. Large resistance removes such free oscillation but does not change the condition $X=0$ of the ideal driven series impedance. State whether a question asks for a step response, free decay or sinusoidal steady state.

Worked method

An RC circuit switches to fixed source voltage V_s . Charge conservation and Kirchhoff's law give

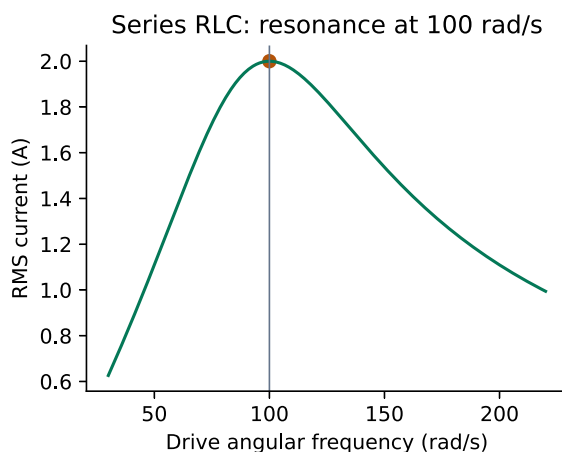
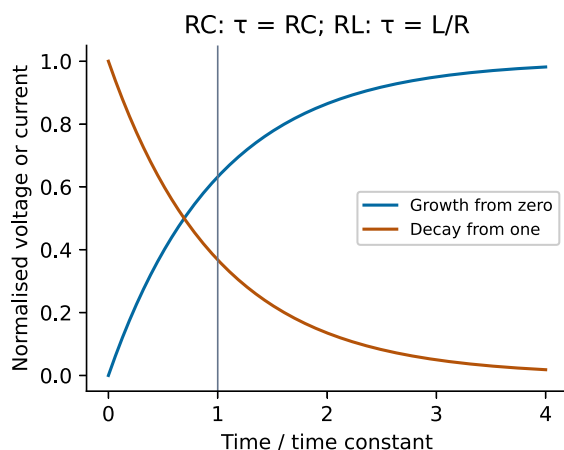
$$RC\dot{V}_C + V_C = V_s, \quad V_C(t) = V_s + [V_C(0) - V_s]e^{-t/(RC)}.$$

For $R = 2$ kilohms, $C = 1$ millifarad, $V_s = 12$ V and initial voltage zero,

$$\tau = RC = (2000\ \Omega)(0.001\ \text{F}) = 2.0\ \text{s}.$$

$$V_C(\tau) = V_s(1 - e^{-1}) = (12\ \text{V})(1 - e^{-1}) = 7.59\ \text{V}.$$

The time constant controls the remaining difference from the final state.



RC and RL transients, impedance and resonance: original GRE teaching diagram.

Check conditions and vocabulary

RC has $\tau=RC$, while RL has $\tau=L/R$. Do not confuse current amplitude with rms current, or resonance of a driven impedance with damped free-oscillation frequency.

time constant: The exponential response scale, RC for an RC circuit and L/R for an RL circuit.

impedance: The complex voltage-to-current phasor ratio in sinusoidal steady state.