

Hamiltonian mechanics and astrophysical scaling

GRE · GRE Subject Test · GRE Physics

Energy methods connect a spring oscillator and an orbit, but their coordinates and physical assumptions differ.

Prerequisites: 26, 41.

- Construct a simple Hamiltonian
- Use canonical equations
- Apply gravitational and radiative scaling

Choose the system and model

The canonical momentum is $p = \partial L / \partial \dot{q}$. The Legendre transform $H = p\dot{q} - L$ gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

Use the governing relation

Hamilton's equations are $\dot{q} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial q$. For an ordinary oscillator $H = p^2 / (2m) + kq^2 / 2$, these reproduce Newton's equation.

Apply the conditions

For a circular gravitational orbit $v^2 = GM/r$ and $\text{period}^2 = 4\pi^2 r^3 / (GM)$. The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass m , angular momentum magnitude is $mr v = m \sqrt{GM r}$, so identical satellites have L proportional to $\sqrt{r}$. A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

Check the conclusion

Luminosity and received flux obey $F = L / (4\pi d^2)$ for isotropic radiation. A black-body has $L = 4\pi R^2 \sigma T^4$; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

Worked method

For a regular Lagrangian $L = m\dot{q}^2/2 - kq^2/2$, canonical momentum is

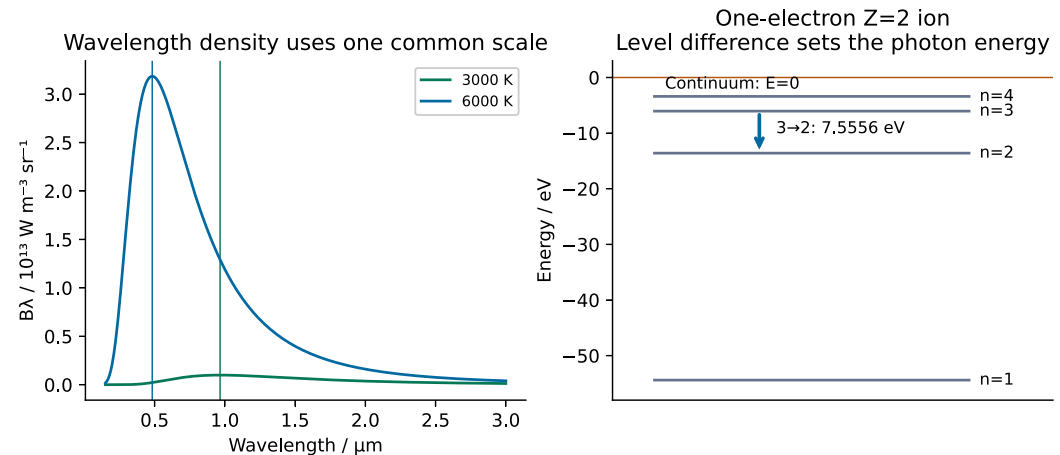
$$p = \partial L / \partial \dot{q} = m\dot{q}, \quad \dot{q} = p/m.$$

Substitute the velocity into the Legendre transform:

$$H = p\dot{q} - L = p^2/(2m) + kq^2/2.$$

$$\dot{q} = \partial H / \partial p = p/m, \quad \dot{p} = -\partial H / \partial q = -kq.$$

Combining these gives $\ddot{q} = -(k/m)q$. The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.



Hamiltonian mechanics and astrophysical scaling: original GRE teaching diagram.

Check conditions and vocabulary

A Hamiltonian equals total mechanical energy only under the relevant system assumptions; do not infer this universally from its name.

canonical momentum: The derivative of the Lagrangian with respect to a generalised velocity.

luminosity: Total emitted power, distinct from flux at an observer.