

# Hamiltonian mechanics and astrophysical scaling

GRE · GRE Subject Test · GRE Physics

Energy methods connect a spring oscillator and an orbit, but their coordinates and physical assumptions differ.

Prerequisites: 26, 41.

- Construct a simple Hamiltonian
- Use canonical equations
- Apply gravitational and radiative scaling

## Choose the system and model

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The canonical momentum is  $p = \partial L / \partial \dot{q}$ . The Legendre transform  $H = p \dot{q} - L$  gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

## Use the governing relation

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Hamilton's equations are  $\dot{q} = \partial H / \partial p$  and  $\dot{p} = -\partial H / \partial q$ . For an ordinary oscillator  $H = p^2 / (2m) + kq^2 / 2$ , these reproduce Newton's equation.

## Apply the conditions

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For a circular gravitational orbit  $v^2 = GM/r$  and  $\text{period}^2 = 4\pi^2 r^3 / (GM)$ . The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass  $m$ , angular momentum magnitude is  $mr v = m \sqrt{GM r}$ , so identical satellites have  $L$  proportional to  $\sqrt{r}$ . A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

## Check the conclusion

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Luminosity and received flux obey  $F = L / (4\pi d^2)$  for isotropic radiation. A black-body has  $L = 4\pi R^2 \sigma T^4$ ; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

## Worked method

For a regular Lagrangian  $L = m\dot{q}^2/2 - kq^2/2$ , canonical momentum is

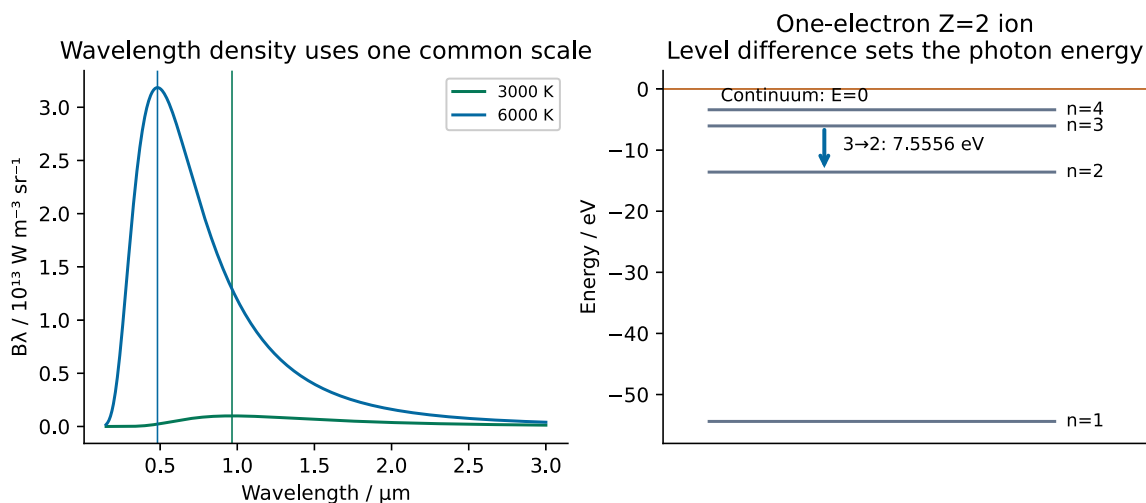
$$p = \partial L / \partial \dot{q} = m\dot{q}, \quad \dot{q} = p/m.$$

Substitute the velocity into the Legendre transform:

$$H = p\dot{q} - L = p^2/(2m) + kq^2/2.$$

$$\dot{q} = \partial H / \partial p = p/m, \quad \dot{p} = -\partial H / \partial q = -kq.$$

Combining these gives  $\ddot{q} = -(k/m)q$ . The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.



*Hamiltonian mechanics and astrophysical scaling: original GRE teaching diagram.*

## Check conditions and vocabulary

A Hamiltonian equals total mechanical energy only under the relevant system assumptions; do not infer this universally from its name.

**canonical momentum:** The derivative of the Lagrangian with respect to a generalised velocity.

**luminosity:** Total emitted power, distinct from flux at an observer.