

47.1 Coordinate operators, separated modes and stable updates

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS separation of variables • stability condition

Objective

Use the divergence and Laplacian in the coordinate system selected by the symmetry; Solve a simple separated boundary-value mode problem; Check convergence, units and stability of an original numerical update. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

Spherical radial operators: $\nabla^2 f = f'' + (2/r)f'$ and $\nabla \cdot (A_r \hat{r}) = (1/r^2)d(r^2 A_r)/dr$. Cartesian forms do not apply to radial dependence.

Separation $u = X(x)T(t)$ with fixed ends gives $X = \sin(n\pi x/L)$ and $f_n = nc/(2L)$; enforce boundary conditions before summing modes.

Explicit Euler for $y' = -\lambda y$ is stable only for $h \leq 2/\lambda$; a negative factor with magnitude above 1 diverges with alternating sign.

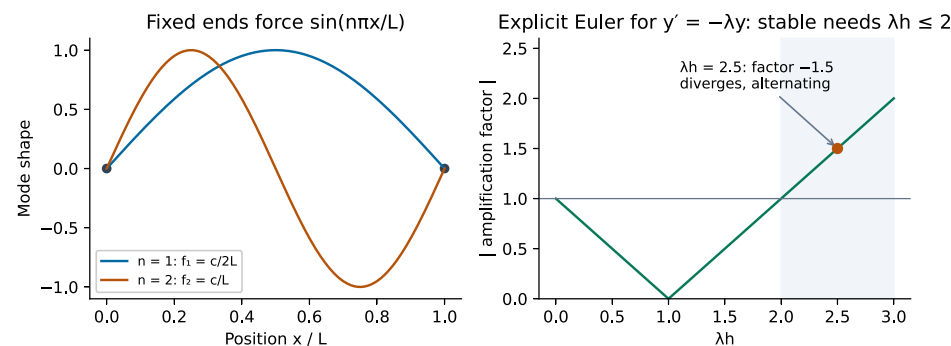
Trust a numerical result only after unit, refinement and limiting-case checks; first-order Euler halves its error with the step.

■ Worked method

A spherical radial field has divergence

$$\nabla \cdot (A_r \hat{r}) = r^{-2} \frac{d}{dr} (r^2 A_r).$$

For $A_r = Cr$, it gives $3C$, not C . Fixed string ends require $X(0) = X(L) = 0$, hence $X_n = \sin(n\pi x/L)$ with positive integer n . Euler for $\dot{y} = -\lambda y$ gives $y_{n+1} = (1 - \lambda h)y_n$. For positive h , bounded absolute stability permits $h \leq 2/\lambda$; decay to zero requires the strict inequality $0 < h < 2/\lambda$. At the upper endpoint the numerical solution alternates without decay. Stability does not establish accuracy.



2 Practice

2.1 Choose one answer. For a radial function $f(r)$ in spherical coordinates, the Laplacian is: [1]

- A f''
- B $f'' + (2/r)f'$
- C $f'' + (1/r)f'$
- D $f'' - (2/r)f'$
- E $(1/r)f''$

2.2 Choose one answer. The explicit Euler update $y_{n+1} = (1 - 10 \times 0.25)y_n$ for $y' = -10y$: [1]

- A converges monotonically
- B converges with decaying oscillation
- C diverges without sign change
- D diverges with alternating sign
- E stays constant

2.3 A string fixed at both ends has $L = 0.5$ m and wave speed 100 m/s. Its fundamental frequency is _____ Hz. Show your relation and units. [2]

[2]

[5]