

39.1 Finite-well bound states and hydrogenic quantum numbers

Name: _____ Class: _____ Date: _____

Total: 11 marks**KEY WORDS** bound state • radial probability density

Objective

Match decaying finite-well states and derive parity-dependent quantisation conditions; Interpret dimensionless bound-state roots and distinguish confinement from scattering; Use hydrogenic quantum numbers, angular momentum and radial probability measures. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For $V = -V_0$ inside $|x| < a$ and zero outside, $-V_0 < E < 0$. Match oscillatory interior to decaying exterior with continuous ψ and ψ' . Even: $k \tan ka = \kappa$; odd: $-k \cot ka = \kappa$.

$z = ka$, $\rho = a\sqrt{(2mV_0)/\hbar^2}$; $\kappa a = \sqrt{(\rho^2 - z^2)}$, $E/V_0 = z^2/\rho^2 - 1$. Solve on valid parity intervals. At $\rho = 1$, $z = \cos z$ gives $z \approx 0.739085$ and $E/V_0 \approx -0.453753$; only the even ground is bound.

For an even state, tails join $A \cos(kx)$ as $A \cos(ka)e^{-\kappa(|x|-a)}$. Integrate inside and both tails for normalisation and outside probability. Bound and continuum scattering states use different asymptotic conditions.

Hydrogenic n, l, m_l obey $l < n$ and $|m_l| \leq l$. $L^2 = l(l+1)\hbar^2$, $L_z = m_l\hbar$; energies $\approx -13.6Z^2/n^2$ eV. Radial density includes r^2 : hydrogen 1s peaks at a_0 and has mean $3a_0/2$, distinct from point density.

■ Worked method

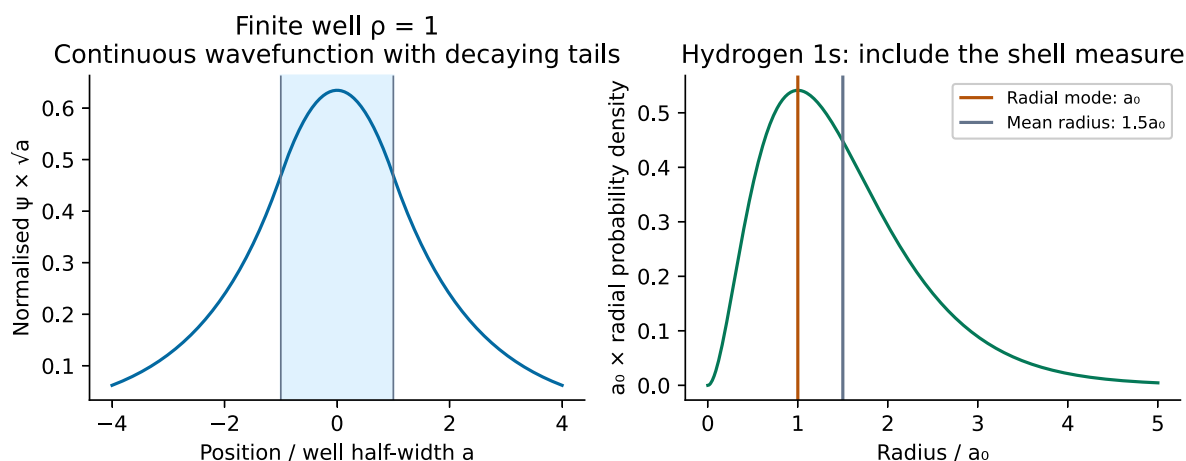
A finite well has $V = -V_0$ for $|x| < a$ and zero outside. A bound energy lies strictly between $-V_0$ and zero. Define

$$k = \sqrt{2m(E + V_0)}/\hbar, \quad \kappa = \sqrt{-2mE}/\hbar.$$

Even parity gives an interior cosine and decaying tails. Matching value and derivative at a gives

$$k \tan(ka) = \kappa.$$

Odd parity instead gives $-k \cot(ka) = \kappa$. Solve only within valid parity intervals; a root at a tangent pole is not a bound state.



2 Practice

2.1 Choose one answer. For a finite square well with equal mass on both sides and no delta interaction, boundary matching requires: [1]

- **A** Only $\psi=0$ at the edge
- **B** Continuous ψ and ψ'
- **C** Continuous ψ' but arbitrary ψ
- **D** Growing tails at infinity
- **E** The energy equals the wall height

2.2 Choose one answer. For a hydrogenic orbital $n=4$, which pair (l, m_l) is allowed? [1]

- **A** $(4,0)$
- **B** $(3,4)$
- **C** $(2,-2)$
- **D** $(-1,0)$
- **E** $(1,2)$

2.3 For $n=4$ in the nonrelativistic Coulomb model, spatial degeneracy before spin is _____. Show your relation and units. [2]

Solutions

2.1 B Both wavefunction and first derivative are continuous at a finite step under these assumptions.

Compare the rejected options: Only $\psi=0$ at the edge; Continuous ψ' but arbitrary ψ ; Growing tails at infinity; The energy equals the wall height. Use the stated model and conditions.

2.2 C l can be 0 through 3, and $|m_l|$ cannot exceed l .

Compare the rejected options: (4,0); (3,4); (-1,0); (1,2). Use the stated model and conditions.

2.3 16. $\sum_{l=0}^3 (2l+1) = 1+3+5+7=16$.

2.4 True. Its external exponential tails are nonzero near the edges but integrable for $\kappa > 0$.

3.1 Interior norm is $A^2[a + \sin(2ka)/(2k)]$. Both tails contribute $A^2 \cos^2(ka)/\kappa$. Thus $A^{-2} = a + \sin(2ka)/(2k) + \cos^2(ka)/\kappa$ and $P_{out} = [\cos^2(ka)/\kappa]/[a + \sin(2ka)/(2k) + \cos^2(ka)/\kappa]$. Bound states require κ positive; a nondecaying threshold tail is not square integrable. Award interior integral, both tails, A , probability and threshold. The matching equation still applies.