

37.1 Weak perturbations and degenerate subspaces

Name: _____ Class: _____ Date: _____

Total: 11 marks**KEY WORDS** first-order energy shift

- degenerate perturbation theory

Objective

Calculate nondegenerate first-order energy shifts as normalised expectation values; Use symmetry and matrix elements to assess weak-coupling limits; Diagonalise the perturbation within an exactly degenerate subspace . Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For $H = H_0 + \lambda W$ and normalised nondegenerate $|n\rangle$, first-order shift is $\lambda \langle n|W|n\rangle$. A weak γx potential in $0 < x < L$ gives $\gamma L/2$, not the potential at a chosen point.

Centred odd perturbations have zero diagonal expectation for a parity state. A centred quadratic shift is $\eta L^2[1/12 - 1/(2\pi^2 n^2)]$. Zero first order does not imply an unchanged exact state.

Weak mixing needs $|\lambda W_{mn}|$ small compared with $|E_n^0 - E_m^0|$. Second-order terms use squared couplings divided by signed gaps; the ground-state denominators are negative.

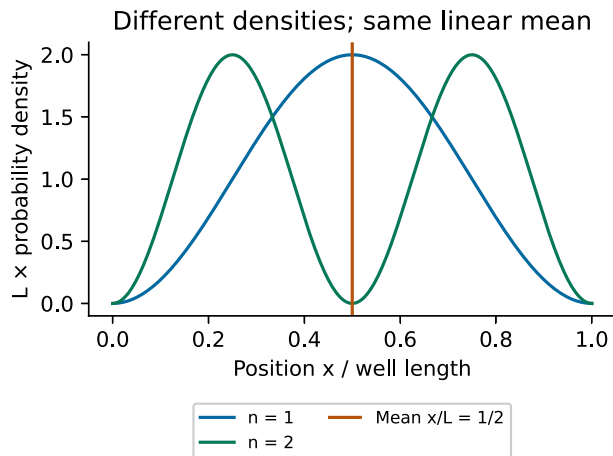
For exact degeneracy, diagonalise the perturbation matrix within the subspace. $\epsilon[[2,1],[1,2]]$ has shifts ϵ and 3ϵ , from antisymmetric/symmetric combinations. Diagonal entries alone are insufficient.

■ Worked method

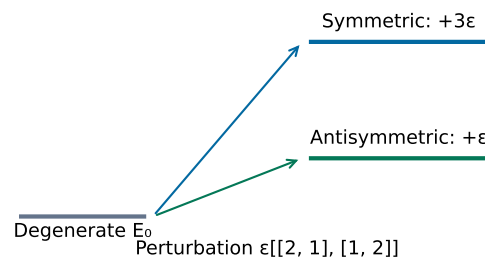
A degenerate subspace requires matrix diagonalisation before first-order shifts. Suppose the perturbation restricted to two states is $\epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$$\det(W - wI) = (2\epsilon - w)^2 - \epsilon^2 = 0.$$

Shifts are epsilon and 3 epsilon. Normalised states are $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$. Reading only the diagonal entries would miss the splitting. Couplings to other levels must remain weak relative to their energy gaps.



Diagonalise the complete degenerate block



2 Practice

2.1 In $0 < x < L$, a weak perturbation $\gamma(x - L/2)$ acts on an unperturbed infinite-well eigenstate. Its first-order energy shift is: [1]

- A. γL
- B. $\gamma L/2$
- C. 0
- D. $-\gamma L/2$
- E. Undefined for every n

2.2 A two-state degenerate block has perturbation $\epsilon[[0, 1], [1, 0]]$, $\epsilon > 0$. First-order shifts are: [1]

- A. 0 and 0
- B. ϵ and ϵ
- C. $-\epsilon$ and $+\epsilon$
- D. 0 and 2ϵ
- E. -2ϵ and 2ϵ

2.3 For a 4 nm well and weak γx with $\gamma = 0.02$ eV/nm, first-order shift is _____ eV. Show your relation and units. [2]

Solutions

2.1 Answer: C. The density is symmetric about $L/2$ while the perturbation is odd, so the integral cancels.

Compare the rejected options: γL ; $\gamma L/2$; $-\gamma L/2$; Undefined for every n . Use the stated model and conditions.

2.2 Answer: C. The matrix eigenvalues are $\pm\varepsilon$. Its vanishing diagonal does not mean vanishing splitting.

Compare the rejected options: 0 and 0; ε and ε ; 0 and 2ε ; -2ε and 2ε . Use the stated model and conditions.

2.3 0.04. $\gamma\langle x \rangle = 0.02 \cdot 4/2 = 0.04$ eV.

2.4 False. Off-diagonal couplings can contribute at second order even when the diagonal expectation cancels.

3.1 Characteristic equation is $E^2 - \Delta E - \epsilon^2 = 0$. $E_{\pm} = (\Delta \pm \sqrt{\Delta^2 + 4\epsilon^2})/2$. Expand the lower root: $E_- \approx -\epsilon^2/\Delta$. The first-order shift is zero, but off-diagonal mixing produces a negative second-order shift. Award equation, roots, expansion, sign and mixing/validity. At $\Delta = 0$ use degenerate theory instead.