

36.1 Quantum occupations and limits of equipartition

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS mean occupation • equipartition

Objective

Compute mean occupation per complete quantum state for fermions and bosons; Count allowed identical-particle occupations and identify the classical dilute limit; Compare classical quadratic-mode heat capacity with a quantum oscillator response. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

One complete state: $n_F = 1/(\hat{x} + 1)$, $n_B = 1/(\hat{x} - 1)$, $x = (\epsilon - \mu)/(k_B T)$. A g -fold level has total g n. Fermionic exclusion is per complete state; photons at equilibrium have $\mu = 0$.

Two identical fermions in four complete states: $\text{choose}(4, 2) = 6$; two bosons: $\text{choose}(5, 2) = 10$; two labelled particles: 16. Spin can distinguish complete states in the same orbital.

For $\hat{x} \gg 1$, both quantum occupations approach $\hat{e}^{-x}$. At $x = \ln 4$: fermion $1/5$, boson $1/3$, classical $1/4$. Particle count alone is not a classical-limit condition.

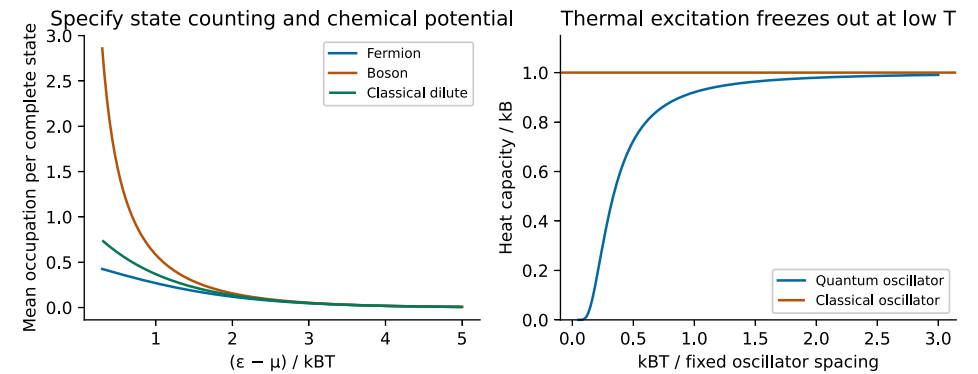
Classical quadratic terms each contribute $k_B T/2$. One 1D oscillator gives $C = k_B$; quantum $C/k_B = x^2 \hat{e}^{-x} / (\hat{e}^{-x} - 1)^2$ tends to one at high T and zero at low T . A fixed zero-point offset contributes no heat capacity.

■ Worked method

Occupation number refers to one complete single-particle state. For $x = (\epsilon - \mu)/(k_B T) = \ln 4$,

$$\bar{n}_F = (e^x + 1)^{-1} = 1/5, \quad \bar{n}_B = (e^x - 1)^{-1} = 1/3.$$

The classical dilute approximation gives $e^{-x} = 1/4$; it is not exact here. A g -fold level has mean total occupation $g\bar{n}$, so a fermionic level can contain more than one particle while each complete state still obeys exclusion.



2 Practice

2.1 Choose one answer. At $(\epsilon - \mu)/(k_B T) = \ln 3$, the mean occupation of one fermionic state is: [1]

- A $1/2$
- B $1/3$
- C $1/4$
- D $3/4$
- E 3

2.2 Choose one answer. Two identical fermions can occupy six distinct complete states. How many occupation patterns exist? [1]

- A 6
- B 12
- C 15
- D 21
- E 36

2.3 At $x = \ln 5$, one bosonic state has mean occupation _____ (give an exact decimal). Show your relation and units. [2]

[2]

[4]

Solutions

2.1 C $n_F = 1/(3+1) = 1/4$; bosonic mean would be $1/(3-1) = 1/2$.

Compare the rejected options: $1/2$; $1/3$; $3/4$; 3 . Use the stated model and conditions.

2.2 C Select the two occupied states: $\text{choose}(6,2) = 15$. Ordered assignments double-count identical fermions.

Compare the rejected options: 6 ; 12 ; 21 ; 36 . Use the stated model and conditions.

2.3 0.25 . $n_B = 1/(5-1) = 1/4 = 0.25$.

2.4 True. Heat capacity differentiates energy with respect to temperature; a fixed additive constant has zero derivative.

3.1 Fermions select three of four states: $\binom{4}{3} = 4$. Bosons distribute three indistinguishable particles among four states: $\binom{3+4-1}{3} = 20$ by stars and bars. Labelled particles each choose any state: $4^3 = 64$. Award each count and exclusion/indistinguishability reasoning. Spin is already included in complete states.