

# 18.1 Fourier symmetry and wave superposition

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 10 marks

KEY WORDS orthogonality • Fourier coefficient

## Objective

Use orthogonality to identify Fourier coefficients; Exploit even and odd parity without discarding the constant term; Relate component amplitudes to interference and physical waveforms. Original public practice; not an ETS paper.

## 1 Key knowledge and worked examples

For a real  $2\pi$ -periodic function,  $f = a_0/2 + \sum [a_n \cos(nx) + b_n \sin(nx)]$ . Coefficients are integrals of  $f$  times the relevant basis function over  $[-\pi, \pi]$ , divided by  $\pi$ . Orthogonality separates components. The mean is  $a_0/2$ , not  $a_0$ .

Even  $f$  about the chosen origin has all  $b_n = 0$  because  $f \sin(nx)$  is odd. Odd  $f$  has zero mean and all  $a_n = 0$ . Shifting the origin can mix sine/cosine coefficients without changing harmonic frequencies. An even nonnegative triangular wave can have cosine terms and a nonzero mean.

For periodic  $|x|$  on  $[-\pi, \pi]$ , integration by parts gives mean  $\pi/2$  and  $a_n = 2[(-1)^n - 1]/(\pi n^2)$ . Thus only odd cosine coefficients survive, at  $-4/(\pi n^2)$ . Its continuous cusp gives  $1/n^2$  decay; discontinuous waveforms need a different convergence analysis.

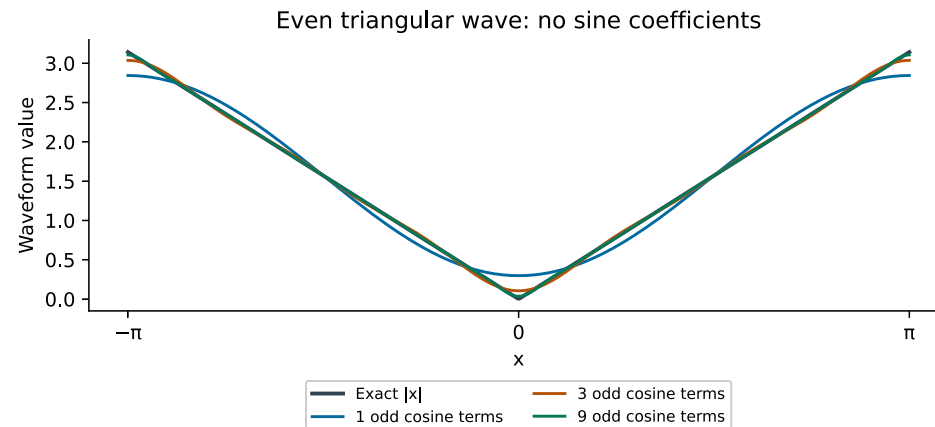
Equal coherent monochromatic fields at phase difference  $\varphi$  give intensity  $2I_0(1 + \cos\varphi)$ :  $4I_0$  in phase and zero at  $\pi$ . Incoherent averaging gives  $2I_0$ . Fourier sums add fields with their phases; a waveform's instantaneous value is not a sum of individual intensities.

### ■ Worked method

For a real periodic function, use  $f = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$ . Even symmetry makes all sine coefficients zero because the integrand is odd. For  $f(x) = |x|$  on  $[-\pi, \pi]$ ,

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) dx = \frac{2[(-1)^n - 1]}{\pi n^2}.$$

Integration by parts supplies the second expression. The mean is  $a_0/2 = \pi/2$ ; only odd cosine harmonics survive. A cusp does not justify dropping the constant term.



## 2 Practice

2.1 Choose one answer. A real even  $2\pi$ -periodic waveform about  $x=0$  necessarily has: [1]

- A All  $a_n = 0$
- B All  $b_n = 0$
- C Zero mean
- D Only its fundamental
- E Equal  $a_n$  and  $b_n$

2.2 Choose one answer. For the periodic extension of  $|x|$ ,  $a_2$  is: [1]

- A  $-4/\pi$
- B  $-1/\pi$
- C 0
- D  $\pi/2$
- E 2

2.3 If  $a_0 = 6$ , the waveform mean is \_\_\_\_\_. Show your relation and units. [2]

[2]

[4]

## Solutions

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**2.1 B** Even times sine is odd, so each symmetric sine-coefficient integral is zero.

Compare the rejected options: All  $a_n=0$ ; Zero mean; Only its fundamental; Equal  $a_n$  and  $b_n$ . Use the stated model and conditions.

**2.2 C**  $a_n=2[(-1)^{n-1}]/(\pi n^2)$ ; even  $n$  gives zero.

Compare the rejected options:  $-4/\pi$ ;  $-1/\pi$ ;  $\pi/2$ ; 2. Use the stated model and conditions.

**2.3 3.** The constant term is  $a_0/2$ .

**2.4 True.** The angle-addition identities mix coefficients at the same frequency.

**3.1** Substitute the shift in  $f \approx \pi/2 - (4/\pi) \cos x$ :  $g \approx \pi/2 - (4/\pi) \sin x$ . Its mean remains  $\pi/2$ . Evenness depends on the chosen origin;  $g$  is not even about zero. Frequencies stay unchanged. Award substitution, harmonic/sign, mean and origin/frequency interpretation.