

Exercise walk-through

36.1

Quantum occupations and limits of equipartition

GRE · GRE Subject Test · GRE Physics

36.1 Quantum occupations and limits of equipartition

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS mean occupation

- equipartition

Objective

Compute mean occupation per complete quantum state for fermions and bosons; Count allowed identical-particle occupations and identify the classical dilute limit; Compare classical quadratic-mode heat capacity with a quantum oscillator response. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

One complete state: $n_F = 1/(\hat{e}^x + 1)$, $n_B = 1/(\hat{e}^x - 1)$, $x = (\varepsilon - \mu)/(kBT)$. A g -fold level has total g n. Fermionic exclusion is per complete state; photons at equilibrium have $\mu = 0$.

Two identical fermions in four complete states: choose(4,2)=6; two bosons: choose(5,2)=10
 two labelled particles: 16. Spin can distinguish complete states in the same orbital.

For $e^{\hat{x}} \gg 1$, both quantum occupations approach $e^{\hat{(-x)}}$. At $x=\ln 4$: fermion 1/5, boson 1/3, classical 1/4. Particle count alone is not a classical-limit condition.

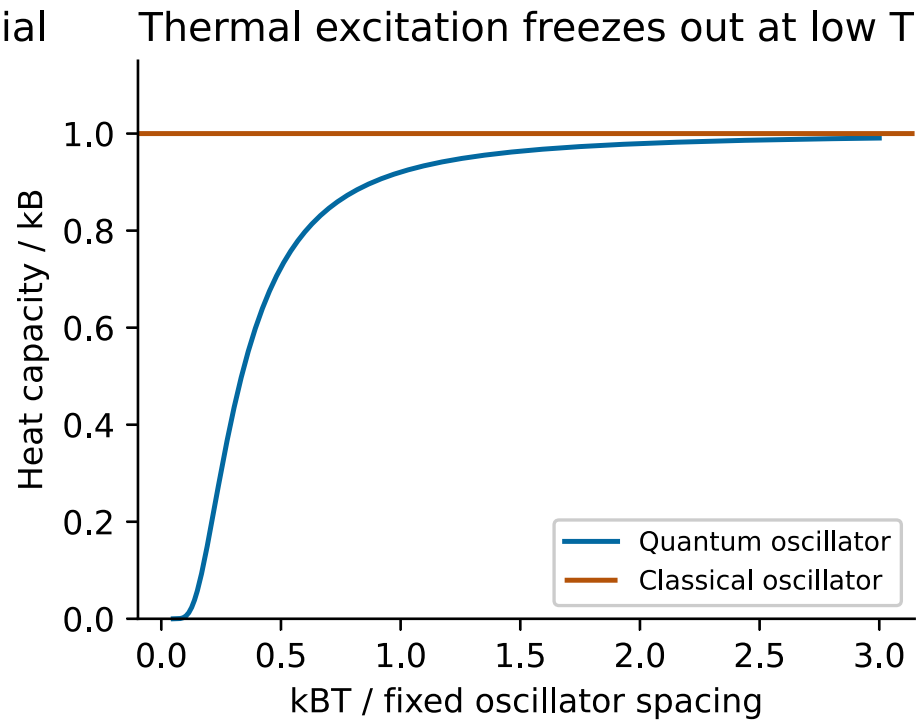
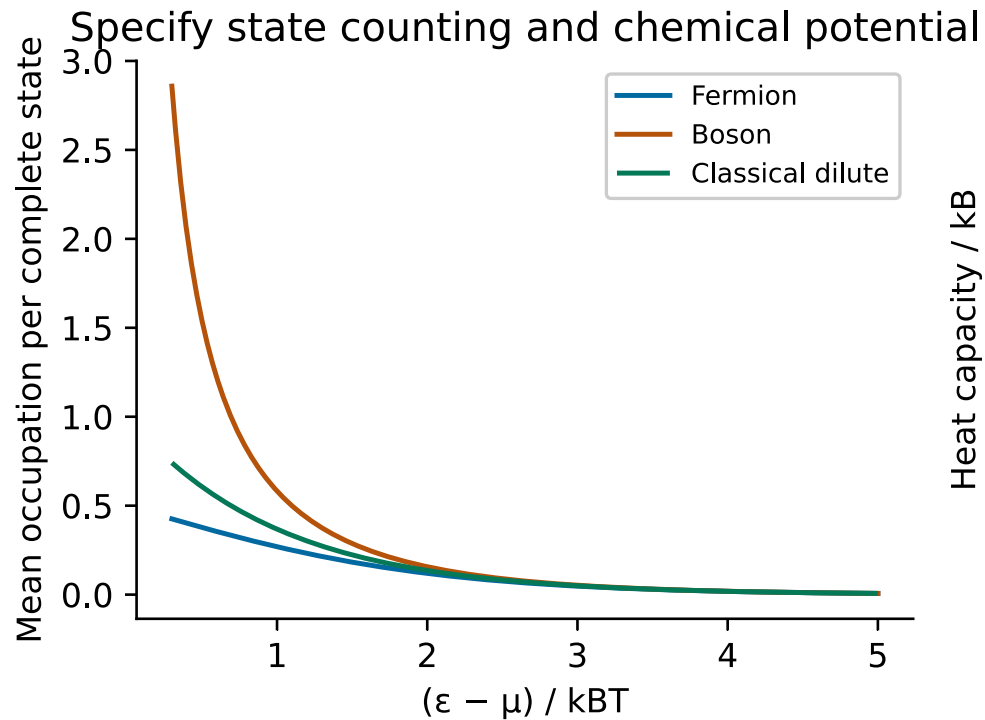
Classical quadratic terms each contribute $k_B T/2$. One 1D oscillator gives $C=k_B$; quantum $C/k_B = x^2 e^{\hat{x}} / (e^{\hat{x}} - 1)^2$ tends to one at high T and zero at low T. A fixed zero-point offset contributes no heat capacity.

■ **Worked method**

Occupation number refers to one complete single-particle state. For $x = (\epsilon - \mu)/(k_B T) = \ln 4$,

$$\bar{n}_F = (e^x + 1)^{-1} = 1/5, \quad \bar{n}_B = (e^x - 1)^{-1} = 1/3.$$

The classical dilute approximation gives $e^{-x} = 1/4$; it is not exact here. A g -fold level has mean total occupation $g\bar{n}$, so a fermionic level can contain more than one particle while each complete state still obeys exclusion.



Exercise 2.1

2.1 Choose one answer. At $(\varepsilon - \mu)/(kBT) = \ln 3$, the mean occupation of one fermionic state is: [1]

- **A** $1/2$
- **B** $1/3$
- **C** $1/4$
- **D** $3/4$
- **E** 3

Exercise 2.1

2.1 Choose one answer. At $(\varepsilon - \mu)/(kBT) = \ln 3$, the mean occupation of one fermionic state is: [1]

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2.1 C $n_F = 1/(3+1) = 1/4$; bosonic mean would be $1/(3-1) = 1/2$.

Compare the rejected options: $1/2$; $1/3$; $3/4$; 3 . Use the stated model and conditions.

Exercises 2.2·2.3

2.2 Choose one answer. Two identical fermions can occupy six distinct complete states. How many occupation patterns exist? [1]

- A 6
- B 12
- C 15
- D 21
- E 36

2.3 At $x=\ln 5$, one bosonic state has mean occupation _____ (give an exact decimal). Show your relation and units. [2]

Exercises 2.2·2.3

2.2 Choose one answer. Two identical fermions can occupy six distinct complete states. How many occupation patterns exist? [1]

- A 6
- B 12
- C 15
- D 21
- E 36

2.2 C Select the two occupied states: $\text{choose}(6,2)=15$. Ordered assignments double-count identical fermions.

Compare the rejected options: 6; 12; 21; 36. Use the stated model and conditions.

2.3 At $x=\ln 5$, one bosonic state has mean occupation _____ (give an exact decimal). Show your relation and units. [2]

2.3 0.25. $n_B = 1/(5-1) = 1/4 = 0.25$.

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: A temperature-independent oscillator zero-point energy $\varepsilon/2$ contributes zero to its heat capacity. [2]

3.1 Three identical particles access four complete states with no energy restriction. Count occupation patterns for fermions and bosons, and compare with three labelled particles. Explain each counting rule. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: A temperature-independent oscillator zero-point energy $\varepsilon/2$ contributes zero to its heat capacity. [2]

2.4 True. Heat capacity differentiates energy with respect to temperature; a fixed additive constant has zero derivative.

3.1 Three identical particles access four complete states with no energy restriction. Count occupation patterns for fermions and bosons, and compare with three labelled particles. Explain each counting rule. [4]

3.1 Fermions select three of four states: $\binom{4}{3} = 4$. Bosons distribute three indistinguish-