

Exercise walk-through

26.1

Constraints, variational equations and cyclic coordinates

GRE · GRE Subject Test · GRE Physics

26.1 Constraints, variational equations and cyclic coordinates

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS holonomic constraint

- cyclic coordinate

Objective

Derive Euler–Lagrange motion after imposing a holonomic constraint ; Use a cyclic coordinate to identify conserved canonical momentum; Linearise a stable equilibrium and check the resulting frequency. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

Use a fixed-length pendulum constraint $x=l \sin\theta$, $y=-l \cos\theta$. Both velocity components give $T=ml^2\dot{\theta}^2/2$; $V=mgl(1-\cos\theta)$. A holonomic constraint relates coordinates, not merely arbitrary velocities.

With ideal constraints, $L=T-V$ and $d(\partial L/\partial \dot{q})/dt - \partial L/\partial q = 0$. The pendulum equation is $ml^2\ddot{\theta} + mgl \sin\theta = 0$. Differentiate all time-dependent factors.

Central motion has $L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$. θ is cyclic, so $mr^2\dot{\theta}$ is constant; $m\ddot{r} = mr\dot{\theta}^2 - V'(r)$. Conserved angular momentum does not imply constant angular speed.

Near a stable equilibrium, $\omega^2 = V''/M$ for the local inertia M . The pendulum approximation $\sin\theta \approx \theta$ requires radians and small amplitude; $\omega = \sqrt{g/l}$. Negative curvature indicates instability.

■ Worked method

For a fixed pendulum length l , use θ from downward vertical: $x = l \sin \theta$, $y = -l \cos \theta$. Both Cartesian velocities give $T = ml^2\dot{\theta}^2/2$ and $V = mgl(1 - \cos \theta)$. The Euler-Lagrange equation — is

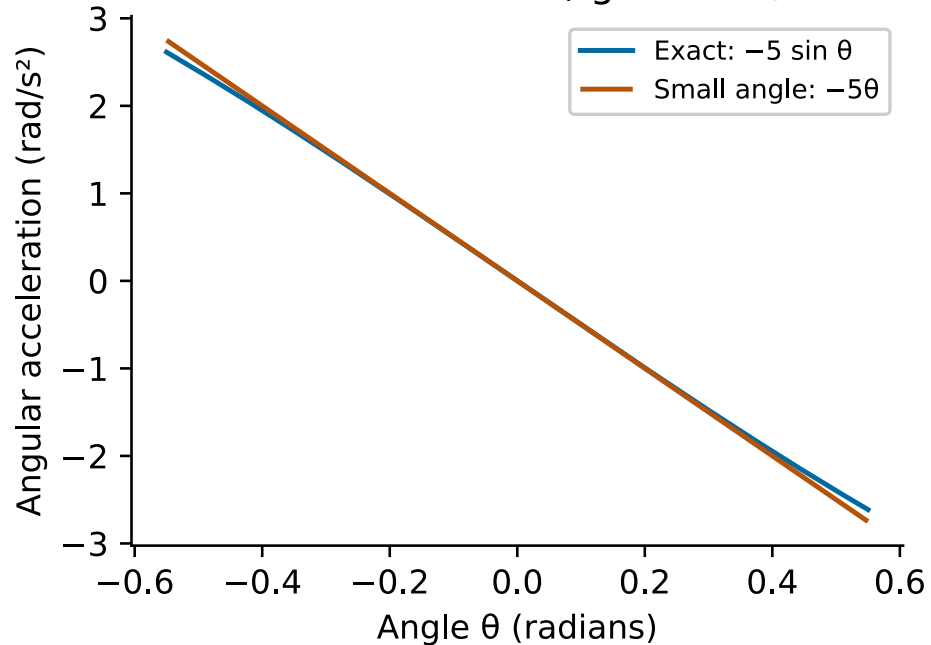
$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin \theta = 0.$$

Divide by ml^2 before linearising:

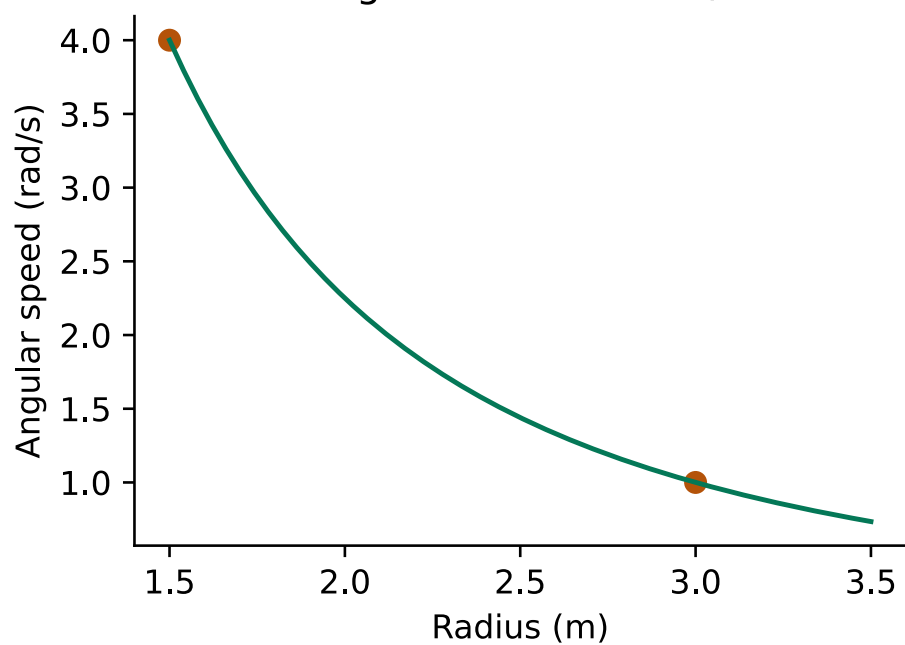
$$\ddot{\theta} + (g/l) \sin \theta = 0, \quad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.

Pendulum $l = 2 \text{ m}$; $g = 10 \text{ m/s}^2$



Constant angular momentum / mass = 9



Exercise 2.1

2.1 Choose one answer. For a pendulum angle θ measured from downward vertical, which kinetic energy uses both Cartesian velocity components? [1]

- **A** $ml^2 \sin^2\theta \dot{\theta}^2/2$
- **B** $ml^2 \cos^2\theta \dot{\theta}^2/2$
- **C** $ml^2\dot{\theta}^2/2$
- **D** $mgl\theta^2/2$
- **E** $ml\dot{\theta}/2$

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2.1 C Adding $\dot{x}^2$ and $\dot{y}^2$ gives $l^2\dot{\theta}^2$.

Compare the rejected options: $ml^2 \sin^2\theta \dot{\theta}^2/2$; $ml^2 \cos^2\theta \dot{\theta}^2/2$; $mgl\theta^2/2$; $ml\dot{\theta}/2$.
Use the stated model and conditions.

Exercises 2.2·2.3

2.2 Choose one answer. In $L = m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$, which statement follows from the cyclic angle? [1]

- **A** θ is constant
- **B** $\dot{\theta}$ is constant
- **C** r is constant
- **D** $mr^2\dot{\theta}$ is constant
- **E** Mechanical kinetic energy is constant

2.3 For $V = 9q^2/2$ and local inertia $M = 1$, small-oscillation angular frequency is _____ rad/s. Show your relation and units. [2]

Exercises 2.2-2.3

2.2 Choose one answer. In $L = m(r\dot{\theta}^2 + r^2\ddot{\theta})/2 - V(r)$, which statement follows from the cyclic angle? [1]

- **A** θ is constant
- **B** $\dot{\theta}$ is constant
- **C** r is constant
- **D** $mr^2\dot{\theta}$ is constant
- **E** Mechanical kinetic energy is constant

2.2 D $\partial L / \partial \theta = 0$ gives $d(mr^2\dot{\theta})/dt = 0$; radius and angular speed may change.

Compare the rejected options: θ is constant; $\dot{\theta}$ is constant; r is constant; Mechanical kinetic energy is constant. Use the stated model and conditions.

2.3 For $V = 9q^2/2$ and local inertia $M = 1$, small-oscillation angular frequency is _____ rad/s. Show your relation and units. [2]

2.3 3. $V'' = 9$ and $\sqrt{9/1} = 3$.

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: The small-angle pendulum approximation is unchanged if the angle is inserted numerically in degrees. [2]

3.1 A bead of mass m slides without friction on a horizontal radial slot rotating at prescribed constant angular speed Ω . Use radius r as coordinate to derive its equation and conserved energy-like quantity. Is its ordinary mechanical energy conserved? Explain. [5]

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: The small-angle pendulum approximation is unchanged if the angle is inserted numerically in degrees. [2]

2.4 False. $\sin\theta \approx \theta$ requires radians; a degree value needs conversion.

3.1 A bead of mass m slides without friction on a horizontal radial slot rotating at prescribed constant angular speed Ω . Use radius r as coordinate to derive its equation and conserved energy-like quantity. Is its ordinary mechanical energy conserved? Explain. [5]

3.1 Prescribed rotation gives $L = m(\dot{r}^2 + \Omega^2 r^2)/2$. Euler-Lagrange gives $\ddot{r} = \Omega^2 r$. The