Constraints, variational equations and cyclic coordinates

GRE · GRE Subject Test · GRE Physics

A pendulum tension changes throughout a swing, but its angular equation can be derived without solving for tension.

Prerequisites: 1, 47.

- Derive Euler–Lagrange motion after imposing a holonomic constraint
- Use a cyclic coordinate to identify conserved canonical momentum
- Linearise a stable equilibrium and check the resulting frequency

Choose independent coordinates

A constraint removes an independent coordinate before you form the kinetic energy. For a fixed-length pendulum choose θ from the downward vertical: $x=l\sin\theta$ and $y=-l\cos\theta$. Differentiating both coordinates gives $v^2=l^2\theta\mathrm{dot}^2$, not $l^2\sin^2\theta\ \theta\mathrm{dot}^2$. Thus $T=ml^2\theta\mathrm{dot}^2/2$ and $V=mgl(1-\cos\theta)$, with zero potential at the bottom. The fixed length is a holonomic constraint: it is a relation among coordinates and possibly time. A rolling velocity constraint needs its own analysis; do not automatically treat every constraint as a coordinate substitution.

Derive the equation

For ideal constraints and the usual conservative system, the stationary-action equation is $d/dt(\partial L/\partial q\mathrm{dot})-\partial L/\partial q=0$ with $L=T-V$. The variations vanish at the two time endpoints. A pendulum gives $\partial L/\partial\theta\mathrm{dot}=ml^2\theta\mathrm{dot}$ and $\partial L/\partial\theta=-mgl\sin\theta$, so $ml^2\theta\mathrm{ddot}+mgl\sin\theta=0$. The momentum derivative acts on every time-dependent factor: for a varying radius, $d(mr^2\theta\mathrm{dot})/dt$ contains $2mr\ r\mathrm{dot}\ \theta\mathrm{dot}$. The variational method does not mean mechanical energy is conserved in a time-dependent system.

Find cyclic momentum

For planar central motion, $L=m(r\mathrm{dot}^2+r^2\theta\mathrm{dot}^2)/2-V(r)$. The angle θ is cyclic because L has no explicit θ dependence, even though L depends on θdot. Its canonical momentum $p\theta=mr^2\theta\mathrm{dot}$ is constant. The radial equation is $m\ r\mathrm{ddot}=mr\theta\mathrm{dot}^2-dV/dr$. The first term is part of the coordinate acceleration, not a new outward real force in an inertial frame. Conservation of $p\theta$ makes angular speed increase as r decreases; constant angular speed is a different, externally driven situation.

Linearise with conditions

Near a stable equilibrium q0, expand V to second order and use a constant local inertia M: $V\approx V(q0)+V''(q0)(q-q0)^2/2$. Then the small-displacement frequency is $\mathrm{sqrt}(V''/M)$. For a pendulum $\sin\theta\approx\theta$ in radians, so $\omega=\mathrm{sqrt}(g/l)$; the approximation requires small amplitude. A negative V'' gives instability, not an oscillation with an imaginary measurable frequency. Check the full nonlinear equation first, then state the approximation and compare units: V''/M must have units of inverse time squared.

Worked method

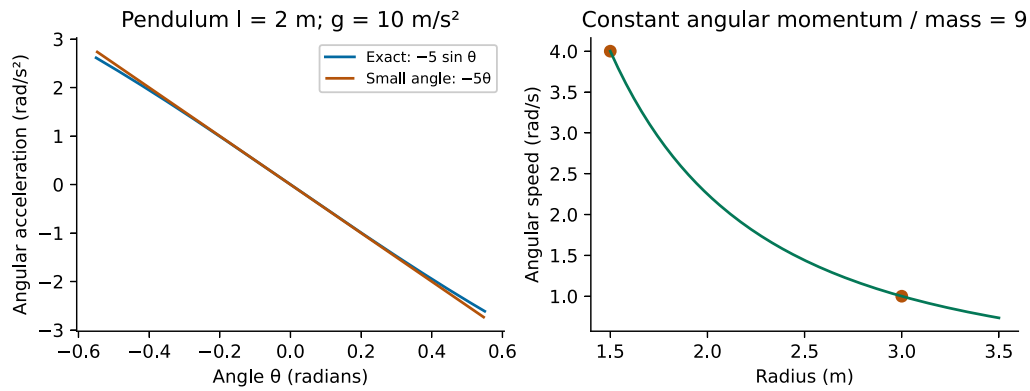
For a fixed pendulum length l, use theta from downward vertical: x = l sin theta, y = -l cos theta. Both Cartesian velocities give $T = ml^2\dot\theta^2/2$ and $V = mgl(1-\cos\theta)$. The Euler-Lagrange equation – is

$$\frac{d}{dt}(ml^2\dot\theta) + mgl\sin\theta = 0.$$

Divide by ml^2 before linearising:

$$\ddot\theta + (g/l)\sin\theta = 0, \qquad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.



Constraints, variational equations and cyclic coordinates: original GRE teaching diagram.

Check conditions and vocabulary

A cyclic coordinate is absent from L itself; its velocity need not be absent. Do not discard the time derivative of a changing r^2 factor.

holonomic constraint: A constraint expressible as a relation among coordinates and possibly time.

cyclic coordinate: A coordinate absent explicitly from the Lagrangian, with conserved canonical momentum under the Euler–Lagrange equation.