9.1 Hamiltonian mechanics and astrophysical scaling

Name: _____ Class: _____ Date: _____ **Total: 11 marks**

KEY WORDS canonical momentum • luminosity

Objective

Construct a simple Hamiltonian; Use canonical equations; Apply gravitational and radiative scaling. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

The canonical momentum is p=∂L/∂qdot. The Legendre transform H=p qdot−L gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

Hamilton's equations are qdot=∂H/∂p and pdot=−∂H/∂q. For an ordinary oscillator $H=p^2/(2m)+kq^2/2$, these reproduce Newton's equation.

For a circular gravitational orbit $v^2=GM/r$ and $period^2=4\pi^2r^3/(GM)$. The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass m, angular momentum magnitude is mrv=m sqrt(GMr), so identical satellites have L proportional to sqrt(r). A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

Luminosity and received flux obey $F=L/(4\pi d^2)$ for isotropic radiation. A blackbody has $L=4\pi R^2\sigma T^4$; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

■ Worked method

For a regular Lagrangian $L = m\dot{q}^2/2 - kq^2/2$, canonical momentum is

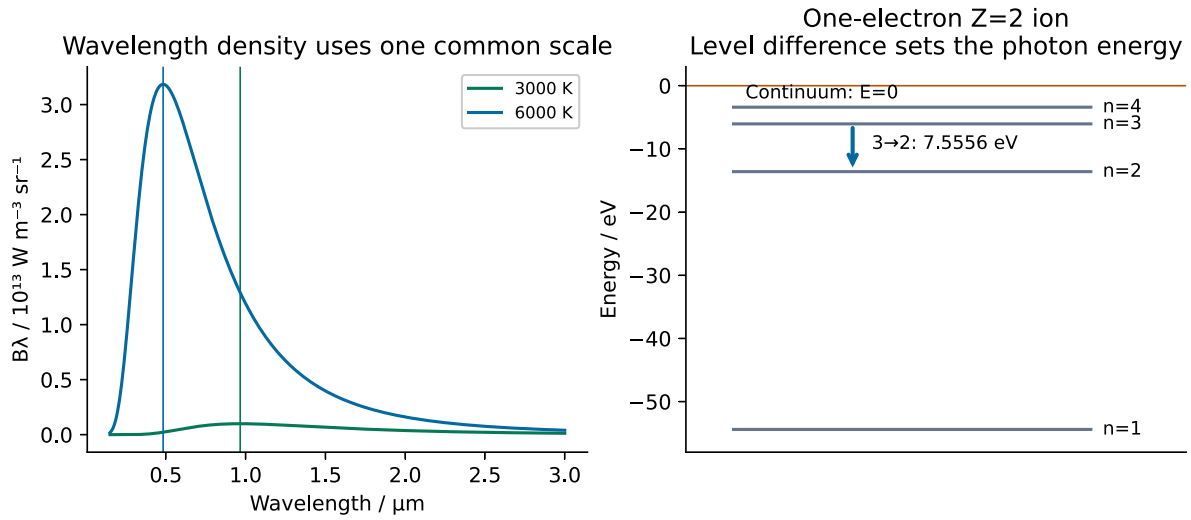
$$p = \partial L/\partial\dot{q} = m\dot{q}, \qquad \dot{q} = p/m.$$

Substitute the velocity into the Legendre transform:

$$H = p\dot{q} - L = p^2/(2m) + kq^2/2.$$

$$\dot{q} = \partial H/\partial p = p/m, \qquad \dot{p} = -\partial H/\partial q = -kq.$$

Combining these gives $\ddot{q} = -(k/m)q$. The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.



2 Practice

2.1 Choose one answer. For $H=p^2/(2m)+kq^2/2$, pdot is: [1]

- **A** kq
- **B** −kq
- **C** p/m

2.2 Choose one answer. Doubling distance changes isotropic flux by factor: [1]

- **A** 2
- **B** 1/2
- **C** 1/4

2.3 For a blackbody of fixed radius, doubling temperature multiplies luminosity by _____. Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: Observed brightness and intrinsic luminosity are identical quantities. [2]

3 Exam-style questions

3.1 An isotropically radiating blackbody star has twice the radius and half the temperature of another. It is three times farther from the observer. Find the luminosity ratio and received-flux ratio, ignoring absorption. Separately, derive Hamilton's equations for $H = p^2/(2m) + aq^4$ with a positive. [5]