

9.1 Hamiltonian mechanics and astrophysical scaling

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS canonical momentum • luminosity

Objective

Construct a simple Hamiltonian; Use canonical equations; Apply gravitational and radiative scaling. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

The canonical momentum is $p = \partial L / \partial \dot{q}$. The Legendre transform $H = p\dot{q} - L$ gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

Hamilton's equations are $\dot{q} = \partial H / \partial p$ and $\dot{p} = -\partial H / \partial q$. For an ordinary oscillator $H = p^2 / (2m) + kq^2 / 2$, these reproduce Newton's equation.

For a circular gravitational orbit $v^2 = GM/r$ and $\text{period}^2 = 4\pi^2 r^3 / (GM)$. The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass m , angular momentum magnitude is $mr v = m \sqrt{GMr}$, so identical satellites have L proportional to $\sqrt{r}$. A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

Luminosity and received flux obey $F = L / (4\pi d^2)$ for isotropic radiation. A blackbody has $L = 4\pi R^2 \sigma T^4$; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

■ Worked method

For a regular Lagrangian $L = m\dot{q}^2/2 - kq^2/2$, canonical momentum is

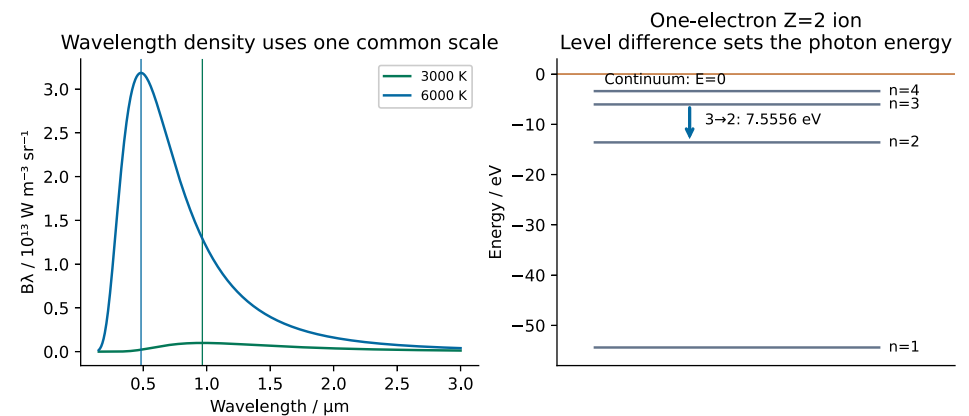
$$p = \partial L / \partial \dot{q} = m\dot{q}, \quad \dot{q} = p/m.$$

Substitute the velocity into the Legendre transform:

$$H = p\dot{q} - L = p^2 / (2m) + kq^2 / 2.$$

$$\dot{q} = \partial H / \partial p = p/m, \quad \dot{p} = -\partial H / \partial q = -kq.$$

Combining these gives $\ddot{q} = -(k/m)q$. The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.



2 Practice

2.1 Choose one answer. For $H = p^2 / (2m) + kq^2 / 2$, $\dot{p}$ is: [1]

- A kq
- B $-kq$
- C p/m

2.2 Choose one answer. Doubling distance changes isotropic flux by factor: [1]

- A 2
- B $1/2$
- C $1/4$

2.3 For a blackbody of fixed radius, doubling temperature multiplies luminosity by _____. Show your relation and units. [2]

[2]

[5]

Solutions

2.1 B Differentiate H with respect to q and negate.

Compare the rejected options: kq ; p/m . Use the stated model and conditions.

2.2 C Inverse-square scaling gives one quarter.

Compare the rejected options: 2 ; $1/2$. Use the stated model and conditions.

2.3 16. Stefan–Boltzmann scaling is T^4 .

2.4 False. Flux also depends on distance and attenuation.

3.1 $L \propto R^2 T^4$ gives $L_2/L_1 = 2^2(1/2)^4 = 1/4$. $F \propto L/d^2$ gives $F_2/F_1 = (1/4)/3^2 = 1/36$. Hamilton's equations give $\dot{q} = p/m$ and $\dot{p} = -4aq^3$. These describe an amplitude-dependent nonlinear oscillator; its restoring force is cubic, not a Hooke-law force with constant nonzero small-oscillation stiffness at zero. Award each ratio, each equation and the nonlinear interpretation.