

# 9.1 Hamiltonian mechanics and astrophysical scaling

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 11 marks

KEY WORDS canonical momentum • luminosity

## Objective

Construct a simple Hamiltonian; Use canonical equations; Apply gravitational and radiative scaling. Original public practice; not an ETS paper.

## 1 Key knowledge and worked examples

The canonical momentum is  $p = \partial L / \partial \dot{q}$ . The Legendre transform  $H = p\dot{q} - L$  gives a Hamiltonian when velocities can be expressed using coordinates and momenta.

Hamilton's equations are  $\dot{q} = \partial H / \partial p$  and  $\dot{p} = -\partial H / \partial q$ . For an ordinary oscillator  $H = p^2 / (2m) + kq^2 / 2$ , these reproduce Newton's equation.

For a circular gravitational orbit  $v^2 = GM/r$  and  $\text{period}^2 = 4\pi^2 r^3 / (GM)$ . The assumptions include a dominant central mass and a circular approximation; elliptical orbits use the semimajor axis in Kepler's law. For a circular satellite of mass  $m$ , angular momentum magnitude is  $mrv = m \sqrt{GMr}$ , so identical satellites have  $L$  proportional to  $\sqrt{r}$ . A radius ratio 9 therefore gives angular-momentum ratio 3, while the period ratio is 27. This distinction follows from the same centripetal-force relation; do not use period scaling for angular momentum.

Luminosity and received flux obey  $F = L / (4\pi d^2)$  for isotropic radiation. A blackbody has  $L = 4\pi R^2 \sigma T^4$ ; its spectral peak shifts inversely with temperature. Distinguish intrinsic luminosity from observed brightness.

### ■ Worked method

For a regular Lagrangian  $L = m\dot{q}^2/2 - kq^2/2$ , canonical momentum is

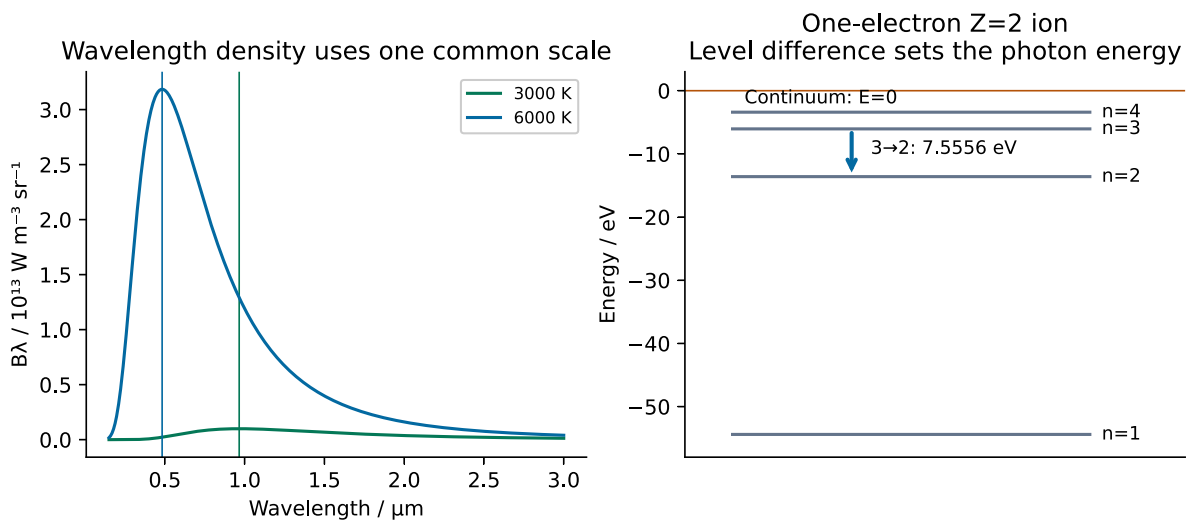
$$p = \partial L / \partial \dot{q} = m\dot{q}, \quad \dot{q} = p/m.$$

Substitute the velocity into the Legendre transform:

$$H = p\dot{q} - L = p^2 / (2m) + kq^2 / 2.$$

$$\dot{q} = \partial H / \partial p = p/m, \quad \dot{p} = -\partial H / \partial q = -kq.$$

Combining these gives  $\ddot{q} = -(k/m)q$ . The inversion of momentum to velocity is required; it cannot be assumed for every constrained or singular Lagrangian.



## 2 Practice

**2.1** Choose one answer. For  $H = p^2/(2m) + kq^2/2$ ,  $p$  dot is: [1]

- **A**  $kq$
- **B**  $-kq$
- **C**  $p/m$

**2.2** Choose one answer. Doubling distance changes isotropic flux by factor: [1]

- **A** 2
- **B**  $1/2$
- **C**  $1/4$

**2.3** For a blackbody of fixed radius, doubling temperature multiplies luminosity by \_\_\_\_\_. Show your relation and units. [2]



## Solutions

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**2.1 B** Differentiate  $H$  with respect to  $q$  and negate.

Compare the rejected options:  $kq$ ;  $p/m$ . Use the stated model and conditions.

**2.2 C** Inverse-square scaling gives one quarter.

Compare the rejected options: 2;  $1/2$ . Use the stated model and conditions.

**2.3 16.** Stefan–Boltzmann scaling is  $T^4$ .

**2.4** False. Flux also depends on distance and attenuation.

**3.1**  $L \propto R^2 T^4$  gives  $L_2/L_1 = 2^2(1/2)^4 = 1/4$ .  $F \propto L/d^2$  gives  $F_2/F_1 = (1/4)/3^2 = 1/36$ . Hamilton's equations give  $\dot{q} = p/m$  and  $\dot{p} = -4aq^3$ . These describe an amplitude-dependent nonlinear oscillator; its restoring force is cubic, not a Hooke-law force with constant nonzero small-oscillation stiffness at zero. Award each ratio, each equation and the nonlinear interpretation.