

47.1 Coordinate operators, separated modes and stable updates

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS separation of variables • stability condition

Objective

Use the divergence and Laplacian in the coordinate system selected by the symmetry; Solve a simple separated boundary-value mode problem; Check convergence, units and stability of an original numerical update. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

Spherical radial operators: $\nabla^2 f = f'' + (2/r)f'$ and $\nabla \cdot (A_r \hat{r}) = (1/r^2)d(r^2 A_r)/dr$. Cartesian forms do not apply to radial dependence.

Separation $u = X(x)T(t)$ with fixed ends gives $X = \sin(n\pi x/L)$ and $f_n = nc/(2L)$; enforce boundary conditions before summing modes.

Explicit Euler for $y' = -\lambda y$ is stable only for $h \leq 2/\lambda$; a negative factor with magnitude above 1 diverges with alternating sign.

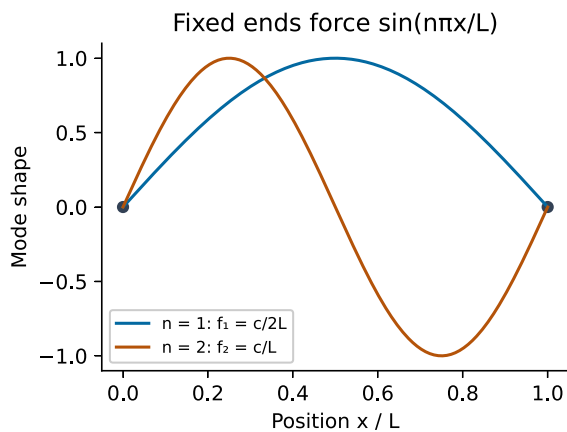
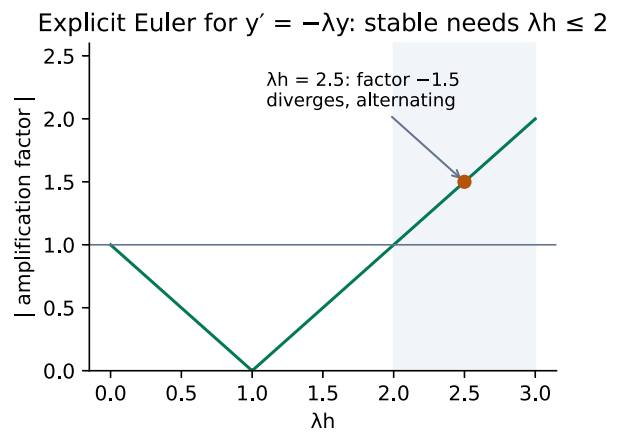
Trust a numerical result only after unit, refinement and limiting-case checks; first-order Euler halves its error with the step.

■ Worked method

A spherical radial field has divergence

$$\nabla \cdot (A_r \hat{r}) = r^{-2} \frac{d}{dr}(r^2 A_r).$$

For $A_r = Cr$, it gives $3C$, not C . Fixed string ends require $X(0) = X(L) = 0$, hence $X_n = \sin(n\pi x/L)$ with positive integer n . Euler for $\dot{y} = -\lambda y$ gives $y_{n+1} = (1 - \lambda h)y_n$. For positive h , bounded absolute stability permits $h \leq 2/\lambda$; decay to zero requires the strict inequality $0 < h < 2/\lambda$. At the upper endpoint the numerical solution alternates without decay. Stability does not establish accuracy.



2 Practice

2.1 Choose one answer. For a radial function $f(r)$ in spherical coordinates, the Laplacian is: [1]

- **A** f''
- **B** $f'' + (2/r)f'$
- **C** $f'' + (1/r)f'$
- **D** $f'' - (2/r)f'$
- **E** $(1/r)f''$

2.2 Choose one answer. The explicit Euler update $y_{n+1} = (1 - 10 \times 0.25)y_n$ for $y' = -10y$: [1]

- **A** converges monotonically
- **B** converges with decaying oscillation
- **C** diverges without sign change
- **D** diverges with alternating sign
- **E** stays constant

2.3 A string fixed at both ends has $L=0.5$ m and wave speed 100 m/s. Its fundamental frequency is _____ Hz. Show your relation and units. [2]

Solutions

2.1 B Spherical radial dependence gives $\nabla^2 f = (1/r^2) d(r^2 f')/dr = f'' + (2/r)f'$.

Compare the rejected options: f'' ; $f'' + (1/r)f'$; $f'' - (2/r)f'$; $(1/r)f''$. Use the stated model and conditions.

2.2 D The factor is -1.5 ; its magnitude exceeds 1 and its sign alternates.

Compare the rejected options: converges monotonically; converges with decaying oscillation; diverges without sign change; stays constant. Use the stated model and conditions.

2.3 100. $f_1 = c/(2L) = 100/1 = 100$ Hz.

2.4 False. Its divergence is zero away from the origin; all source strength sits at the origin as a distribution.

3.1 Explicit factor is $1 - \lambda h$. Nonnegative monotone decay needs $0 < h \leq 1/\lambda$; at equality one step reaches zero, unlike the exact exponential. Implicit Euler solves $y_{n+1} = y_n - \lambda h y_{n+1}$, so its factor is $1/(1 + \lambda h)$, between zero and one for every positive h . It is stable but only first-order accurate; refine h at fixed final time and compare with $e^{-\lambda t}$. Award factors, monotonicity range, endpoint, implicit stability and accuracy/refinement distinction.