

# 37.1 Weak perturbations and degenerate subspaces

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

**Total: 11 marks****KEY WORDS** first-order energy shift

- degenerate perturbation theory

## Objective

Calculate nondegenerate first-order energy shifts as normalised expectation values; Use symmetry and matrix elements to assess weak-coupling limits; Diagonalise the perturbation within an exactly degenerate subspace . Original public practice; not an ETS paper.

## 1 Key knowledge and worked examples

For  $H = H_0 + \lambda W$  and normalised nondegenerate  $|n\rangle$ , first-order shift is  $\lambda \langle n|W|n\rangle$ . A weak  $\gamma x$  potential in  $0 < x < L$  gives  $\gamma L/2$ , not the potential at a chosen point.

Centred odd perturbations have zero diagonal expectation for a parity state. A centred quadratic shift is  $\eta L^2[1/12 - 1/(2\pi^2 n^2)]$ . Zero first order does not imply an unchanged exact state.

Weak mixing needs  $|\lambda W_{mn}|$  small compared with  $|E_n^0 - E_m^0|$ . Second-order terms use squared couplings divided by signed gaps; the ground-state denominators are negative.

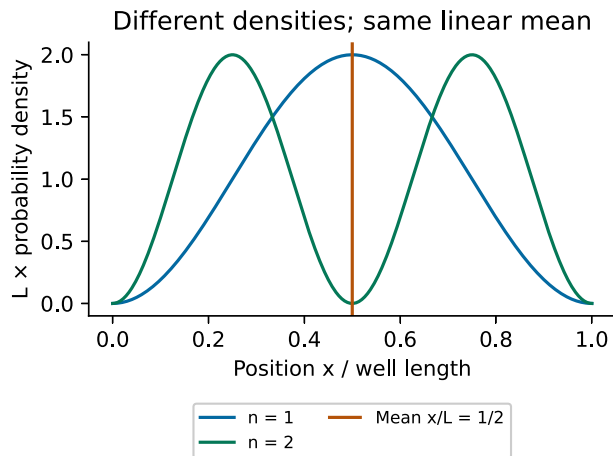
For exact degeneracy, diagonalise the perturbation matrix within the subspace.  $\epsilon[[2,1],[1,2]]$  has shifts  $\epsilon$  and  $3\epsilon$ , from antisymmetric/symmetric combinations. Diagonal entries alone are insufficient.

### ■ Worked method

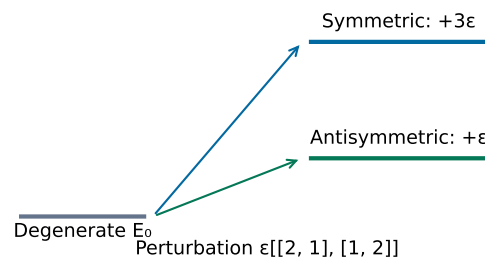
A degenerate subspace requires matrix diagonalisation before first-order shifts. Suppose the perturbation restricted to two states is  $\epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

$$\det(W - wI) = (2\epsilon - w)^2 - \epsilon^2 = 0.$$

Shifts are epsilon and 3 epsilon. Normalised states are  $(1, -1)/\sqrt{2}$  and  $(1, 1)/\sqrt{2}$ . Reading only the diagonal entries would miss the splitting. Couplings to other levels must remain weak relative to their energy gaps.



Diagonalise the complete degenerate block



## 2 Practice

**2.1** Choose one answer. In  $0 < x < L$ , a weak perturbation  $\gamma(x - L/2)$  acts on an unperturbed infinite-well eigenstate. Its first-order energy shift is: [1]

- **A**  $\gamma L$
- **B**  $\gamma L/2$
- **C** 0
- **D**  $-\gamma L/2$
- **E** Undefined for every  $n$

**2.2** Choose one answer. A two-state degenerate block has perturbation  $\epsilon[[0, 1], [1, 0]]$ ,  $\epsilon > 0$ . First-order shifts are: [1]

- **A** 0 and 0
- **B**  $\epsilon$  and  $\epsilon$
- **C**  $-\epsilon$  and  $+\epsilon$
- **D** 0 and  $2\epsilon$
- **E**  $-2\epsilon$  and  $2\epsilon$

**2.3** For a 4 nm well and weak  $\gamma x$  with  $\gamma = 0.02$  eV/nm, first-order shift is \_\_\_\_\_ eV. Show your relation and units. [2]



## Solutions

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**2.1 C** The density is symmetric about  $L/2$  while the perturbation is odd, so the integral cancels.

Compare the rejected options:  $\gamma L$ ;  $\gamma L/2$ ;  $-\gamma L/2$ ; Undefined for every  $n$ . Use the stated model and conditions.

**2.2 C** The matrix eigenvalues are  $\pm\varepsilon$ . Its vanishing diagonal does not mean vanishing splitting.

Compare the rejected options: 0 and 0;  $\varepsilon$  and  $\varepsilon$ ; 0 and  $2\varepsilon$ ;  $-2\varepsilon$  and  $2\varepsilon$ . Use the stated model and conditions.

**2.3** 0.04.  $\gamma\langle x \rangle = 0.02 \cdot 4/2 = 0.04$  eV.

**2.4** False. Off-diagonal couplings can contribute at second order even when the diagonal expectation cancels.

**3.1** Characteristic equation is  $E^2 - \Delta E - \epsilon^2 = 0$ .  $E_{\pm} = (\Delta \pm \sqrt{\Delta^2 + 4\epsilon^2})/2$ . Expand the lower root:  $E_- \approx -\epsilon^2/\Delta$ . The first-order shift is zero, but off-diagonal mixing produces a negative second-order shift. Award equation, roots, expansion, sign and mixing/validity. At  $\Delta = 0$  use degenerate theory instead.