26.1 Constraints, variational equations and cyclic coordinates

Name: _____ Class: _____ Date: _____ **Total: 11 marks**

KEY WORDS holonomic constraint • cyclic coordinate

Objective

Derive Euler–Lagrange motion after imposing a holonomic constraint ; Use a cyclic coordinate to identify conserved canonical momentum; Linearise a stable equilibrium and check the resulting frequency. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

Use a fixed-length pendulum constraint x=l sinθ, y=−l cosθ. Both velocity components give T=ml$^2\theta$dot^2/2; V=mgl(1−cosθ). A holonomic constraint relates coordinates, not merely arbitrary velocities.

With ideal constraints, L=T−V and d(∂L/∂qdot)/dt−∂L/∂q=0. The pendulum equation is ml$^2\theta$ddot+mgl sinθ=0. Differentiate all time-dependent factors.

Central motion has L=m(rdot^2+r$^2\theta$dot^2)/2−V(r). θ is cyclic, so mr$^2\theta$dot is constant; m rddot=mrθdot^2−V′(r). Conserved angular momentum does not imply constant angular speed.

Near a stable equilibrium, ω^2=V″/M for the local inertia M. The pendulum approximation sin$\theta\approx\theta$ requires radians and small amplitude; ω=sqrt(g/l). Negative curvature indicates instability.

■ Worked method

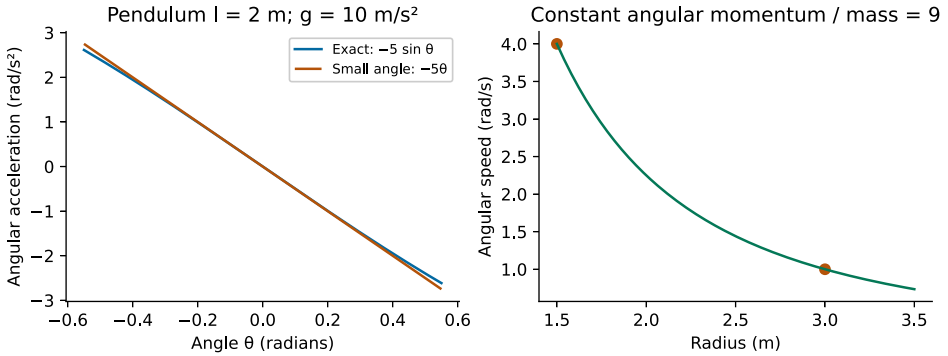
For a fixed pendulum length l, use theta from downward vertical: x = l sin theta, y = -l cos theta. Both Cartesian velocities give $T = ml^2\dot{\theta}^2/2$ and $V = mgl(1 - \cos\theta)$. The Euler-Lagrange equation – is

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0.$$

Divide by ml^2 before linearising:

$$\ddot{\theta} + (g/l)\sin\theta = 0, \qquad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.



2 Practice

2.1 Choose one answer. For a pendulum angle θ measured from downward vertical, which kinetic energy uses both Cartesian velocity components? [1]

- **A** ml^2 sin$^2\theta$ θdot^2/2
- **B** ml^2 cos$^2\theta$ θdot^2/2
- **C** ml$^2\theta$dot^2/2
- **D** mglθ^2/2
- **E** mlθdot/2

2.2 Choose one answer. In L=m(rdot^2+r$^2\theta$dot^2)/2−V(r), which statement follows from the cyclic angle? [1]

- **A** θ is constant
- **B** θdot is constant
- **C** r is constant
- **D** mr$^2\theta$dot is constant
- **E** Mechanical kinetic energy is constant

2.3 For V=9q^2/2 and local inertia M=1, small-oscillation angular frequency is _____ rad/s. Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: The small-angle pendulum approximation is unchanged if the angle is inserted numerically in degrees. [2]

3 Exam-style questions

3.1 A bead of mass m slides without friction on a horizontal radial slot rotating at prescribed constant angular speed Omega. Use radius r as coordinate to derive its equation and conserved energy-like quantity. Is its ordinary mechanical energy conserved? Explain. [5]

Solutions

2.1 C Adding xdot^2 and ydot^2 gives l$^2\theta$dot^2.

Compare the rejected options: ml^2 sin$^2\theta$ θdot$^2/2$; ml^2 cos$^2\theta$ θdot$^2/2$; mgl$\theta^2/2$; mlθdot$/2$. Use the stated model and conditions.

2.2 D ∂L$/\partial\theta$=0 gives d(mr$^2\theta$dot)/dt=0; radius and angular speed may change.

Compare the rejected options: θ is constant; θdot is constant; r is constant; Mechanical kinetic energy is constant. Use the stated model and conditions.

2.3 3. V$''$=9 and sqrt(9/1)=3.

2.4 False. sin$\theta\approx\theta$ requires radians; a degree value needs conversion.

3.1 Prescribed rotation gives $L = m(\dot{r}^2 + \Omega^2 r^2)/2$. Euler-Lagrange gives $\ddot{r} = \Omega^2 r$. The time-independent reduced L has conserved $\dot{r}\,\partial L/\partial\dot{r} - L = m(\dot{r}^2 - \Omega^2 r^2)/2$. Mechanical energy is $m(\dot{r}^2 + \Omega^2 r^2)/2$ and is not generally conserved: the rotation drive supplies work through the constraint. Award L, equation, conserved quantity, mechanical-energy distinction and drive. The bead's angular momentum is not conserved.