

## 26.1 Constraints, variational equations and cyclic coordinates

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 11 marks

KEY WORDS holonomic constraint • cyclic coordinate

### Objective

Derive Euler–Lagrange motion after imposing a holonomic constraint \_\_\_\_\_; Use a cyclic coordinate \_\_\_\_\_ to identify conserved canonical momentum; Linearise a stable equilibrium and check the resulting frequency. Original public practice; not an ETS paper.

### 1 Key knowledge and worked examples

Use a fixed-length pendulum constraint  $x=l \sin\theta$ ,  $y=-l \cos\theta$ . Both velocity components give  $T=ml^2\dot{\theta}^2/2$ ;  $V=mgl(1-\cos\theta)$ . A holonomic constraint relates coordinates, not merely arbitrary velocities.

With ideal constraints,  $L=T-V$  and  $d(\partial L/\partial \dot{q})/dt - \partial L/\partial q = 0$ . The pendulum equation is  $ml^2\ddot{\theta} + mgl \sin\theta = 0$ . Differentiate all time-dependent factors.

Central motion has  $L=m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$ .  $\theta$  is cyclic, so  $mr^2\dot{\theta}$  is constant;  $m\ddot{r} = mr\dot{\theta}^2 - V'(r)$ . Conserved angular momentum does not imply constant angular speed.

Near a stable equilibrium,  $\omega^2 = V''/M$  for the local inertia  $M$ . The pendulum approximation  $\sin\theta \approx \theta$  requires radians and small amplitude;  $\omega = \sqrt{g/l}$ . Negative curvature indicates instability.

#### ■ Worked method

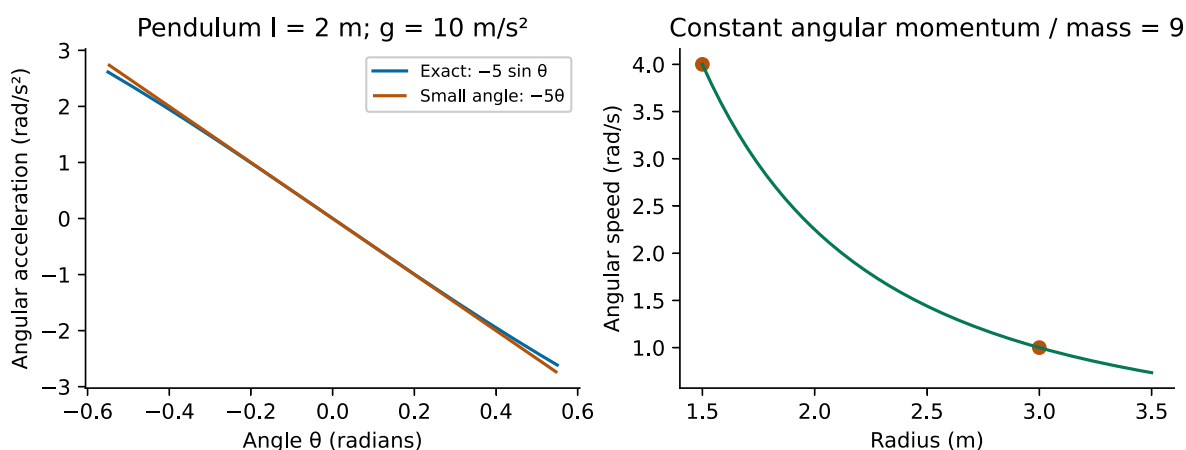
For a fixed pendulum length  $l$ , use  $\theta$  from downward vertical:  $x = l \sin \theta$ ,  $y = -l \cos \theta$ . Both Cartesian velocities give  $T = ml^2\dot{\theta}^2/2$  and  $V = mgl(1 - \cos \theta)$ . The Euler-Lagrange equation \_\_\_\_\_ is

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin \theta = 0.$$

Divide by  $ml^2$  before linearising:

$$\ddot{\theta} + (g/l) \sin \theta = 0, \quad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.



## 2 Practice

**2.1** Choose one answer. For a pendulum angle  $\theta$  measured from downward vertical, which kinetic energy uses both Cartesian velocity components? [1]

- **A**  $ml^2 \sin^2\theta \dot{\theta}^2/2$
- **B**  $ml^2 \cos^2\theta \dot{\theta}^2/2$
- **C**  $ml^2\dot{\theta}^2/2$
- **D**  $mgl\theta^2/2$
- **E**  $ml\dot{\theta}/2$

**2.2** Choose one answer. In  $L=m(\dot{r}^2+r^2\dot{\theta}^2)/2-V(r)$ , which statement follows from the cyclic angle? [1]

- **A**  $\theta$  is constant
- **B**  $\dot{\theta}$  is constant
- **C**  $r$  is constant
- **D**  $mr^2\dot{\theta}$  is constant
- **E** Mechanical kinetic energy is constant

**2.3** For  $V=9q^2/2$  and local inertia  $M=1$ , small-oscillation angular frequency is \_\_\_\_\_ rad/s. Show your relation and units. [2]

---



---



---



## Solutions

---

**2.1 C** Adding  $\dot{x}^2$  and  $\dot{y}^2$  gives  $l^2\dot{\theta}^2$ .

Compare the rejected options:  $ml^2 \sin^2\theta \dot{\theta}^2/2$ ;  $ml^2 \cos^2\theta \dot{\theta}^2/2$ ;  $mg l\theta^2/2$ ;  $ml\dot{\theta}/2$ . Use the stated model and conditions.

**2.2 D**  $\partial L/\partial\theta=0$  gives  $d(mr^2\dot{\theta})/dt=0$ ; radius and angular speed may change.

Compare the rejected options:  $\theta$  is constant;  $\dot{\theta}$  is constant;  $r$  is constant; Mechanical kinetic energy is constant. Use the stated model and conditions.

**2.3 3.**  $V''=9$  and  $\sqrt{9/1}=3$ .

**2.4 False.**  $\sin\theta\approx\theta$  requires radians; a degree value needs conversion.

**3.1** Prescribed rotation gives  $L = m(\dot{r}^2 + \Omega^2 r^2)/2$ . Euler-Lagrange gives  $\ddot{r} = \Omega^2 r$ . The time-independent reduced  $L$  has conserved  $\dot{r} \partial L/\partial \dot{r} - L = m(\dot{r}^2 - \Omega^2 r^2)/2$ . Mechanical energy is  $m(\dot{r}^2 + \Omega^2 r^2)/2$  and is not generally conserved: the rotation drive supplies work through the constraint. Award  $L$ , equation, conserved quantity, mechanical-energy distinction and drive. The bead's angular momentum is not conserved.