22.1 Relativistic lifetime, energy and Doppler shift

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS proper time • rest energy

Objective

Relate proper lifetime to laboratory time and distance; Use invariant energy–momentum and relativistic kinetic energy; Infer longitudinal recession speed from wavelength ratio. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For relative speed v, define β=v/c and γ=1/sqrt(1−β^2). Proper time is measured along the particle's worldline by a clock at rest with it. If its proper mean lifetime is τ0, its laboratory mean lifetime is $\gamma\tau$0 and mean travel distance is v$\gamma\tau$0 at constant speed. The decay is statistical: mean lifetime is not a guaranteed decay time for each particle. At β=0.8, γ=5/3. Compute spacetime events consistently in one frame; proper time and coordinate time are different quantities.

Total energy is E=γmc^2, momentum p=γmv and invariant E^2−p^{2}c^2=m^{2}c^4. When E and pc are expressed in the same energy units, find mc^2=sqrt(E^2−(pc)^2), not E−pc. The nonnegative square root is required for positive rest mass. For E=13 GeV and pc=12 GeV, rest energy is 5 GeV and mass is 5 GeV/c^2. A massless particle has E=pc but need not have zero energy or momentum. These relations concern isolated-particle four-momentum, not classical mv at relativistic speed.

Work accelerating a particle from rest equals kinetic energy K=E−mc^2=(γ−1)mc^2. At β=0.6, γ=1.25 and K=0.25mc^2, while the classical ½mv^2 gives 0.18mc^2. Classical kinetic energy is the low-speed expansion and becomes inaccurate near c. Finite acceleration work increases γ rather than allowing a massive particle to reach or exceed c. Keep total energy, rest energy and kinetic energy distinct when interpreting answer units.

For purely longitudinal relative recession in special relativity, wavelength ratio r=λ_observed/λ_emitted Rearranging gives β=(r^2−1)/(r^2+1). A ratio r=2 gives β=3/5, not c times r−1 from a low-speed approximation. Blueshift uses r<1 and a negative recession parameter under this convention. The formula assumes the shift is entirely kinematic; cosmological expansion or gravitational redshift requires a different model. State the question's stipulated model before interpreting a spectral ratio.

■ Worked method

Total energy and momentum determine invariant rest energy.

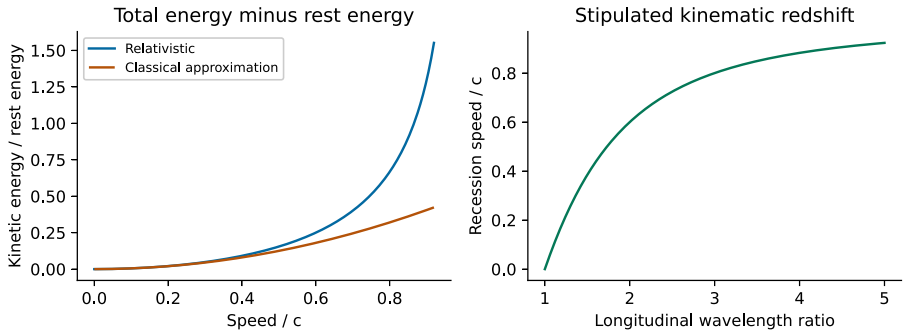
$$E^2 - (pc)^2 = (mc^2)^2, \qquad mc^2 = \sqrt{E^2 - (pc)^2}.$$

For E = 17 GeV and pc = 15 GeV,

$$mc^2 = \sqrt{(17\,\mathrm{GeV})^2 - (15\,\mathrm{GeV})^2} = 8\,\mathrm{GeV}.$$

$$K = E - mc^2 = 17\,\mathrm{GeV} - 8\,\mathrm{GeV} = 9\,\mathrm{GeV}.$$

Rest mass is 8 GeV/c squared. Do not subtract momentum directly from energy.



2 Practice

2.1 Choose one answer. A particle at 0.8c has proper mean lifetime 6 μs. Its laboratory mean lifetime is: [1]

- **A** 3.6 μs
- **B** 4.8 μs
- **C** 6 μs
- **D** 10 μs
- **E** 30 μs

2.2 Choose one answer. For E=10 GeV and pc=6 GeV, rest energy is: [1]

- **A** 4 GeV
- **B** 6 GeV
- **C** 8 GeV
- **D** 10 GeV
- **E** 16 GeV

2.3 At speed 0.6c, $K/(mc^2)$=_____ (decimal). Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: A wavelength redshift ratio of 2 implies superluminal recession under the stipulated longitudinal special-relativity model. [2]

3 Exam-style questions

3.1 Equal-energy photons of energy E collide with directions separated by angle theta. Derive the invariant mass of the pair. Check parallel and head-on limits. [4]

Solutions

2.1 D γ=5/3, so $\gamma\tau$0=10 μs.

Compare the rejected options: 3.6 μs; 4.8 μs; 6 μs; 30 μs. Use the stated model and conditions.

2.2 C sqrt(100−36)=8 GeV.

Compare the rejected options: 4 GeV; 6 GeV; 10 GeV; 16 GeV. Use the stated model and conditions.

2.3 0.25. γ=1.25, so γ−1=0.25.

2.4 False. β=(4−1)/(4+1)=0.6.

3.1 Total energy is 2E; total momentum magnitude is $2(E/c)|\cos(\theta/2)|$. $M^2c^4 = 4E^2 - 4E^2\cos^2(\theta/2) = 4E^2\sin^2(\theta/2)$. Thus $Mc^2 = 2E\sin(\theta/2)$ for $0 \leq \theta \leq \pi$. Parallel photons have zero pair invariant mass; head-on photons have $Mc^2 = 2E$. Award vector momentum, invariant, result and limits. Each photon remains massless.