

20.1 Speed distributions and statistical ensembles

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS probability density • partition function

Objective

Normalise continuous speed densities and interpret their units; Distinguish vector means, speed means and most probable speeds; Use Boltzmann weights with degeneracy and a partition function . Original public practice; not an ETS paper.

1 Key knowledge and worked examples

A speed probability density f has $\int f(v)dv=1$ and inverse-speed units. A count density P has area N and $f=P/N$. A trapezoid rising over v_0 , flat over $2v_0$ and falling over v_0 has area $3av_0$ for height a . Normalise by area, not height.

Without bulk flow an isotropic gas has zero mean velocity but positive mean speed. Maxwell speed density is proportional to $v^2 \exp[-mv^2/(2k_B T)]$. Mode, mean and rms speeds are $\sqrt{2k_B T/m}$, $\sqrt{8k_B T/(\pi m)}$ and $\sqrt{3k_B T/m}$. The v^2 factor makes speed density zero at $v=0$.

For a continuous distribution, an exact single speed has probability zero; intervals have nonzero probabilities. This does not exclude arbitrarily slow particles or a detector's finite-width zero bin. Characteristic speeds scale as $\sqrt{T/m}$, not T .

Canonical state weights are $\exp(-E/(k_B T))$; divide by their sum Z . Level probability includes degeneracy g : $g \exp(-E/(k_B T))/Z$. Ground degeneracy 1 and excited degeneracy 2 at gap $k_B T \ln 2$ give equal level weights. Classical Maxwell-Boltzmann differs from quantum Bose/Fermi occupations.

■ Worked method

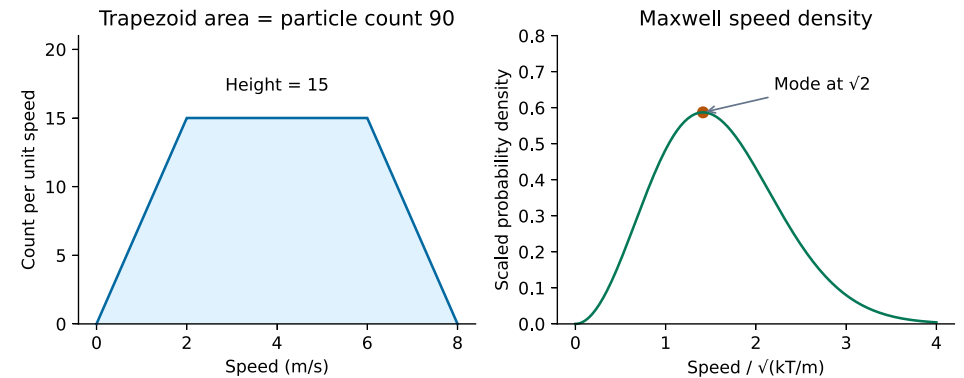
A canonical ensemble has ground degeneracy 1 and excited degeneracy g at gap Δ .

$$Z = 1 + g e^{-\Delta/(k_B T)}, \quad P_e = \frac{g e^{-\Delta/(k_B T)}}{Z}.$$

With $g = 4$ and $\Delta = k_B T \ln 2$,

$$P_e = \frac{4e^{-\ln 2}}{1 + 4e^{-\ln 2}} = \frac{2}{3}.$$

This is probability for the entire excited level. Each of its four states has probability $1/6$.



2 Practice

2.1 Choose one answer. For an isotropic equilibrium ideal gas without bulk flow, which is correct? [1]

- A Mean velocity and mean speed are both zero
- B Mean velocity is zero but mean speed is positive
- C The most probable speed is zero
- D All particles have the rms speed
- E Speed density has no units

2.2 Choose one answer. Ground degeneracy 1, excited degeneracy 3, gap $k_B T \ln 3$. The excited-level probability is: [1]

- A $1/4$
- B $1/3$
- C $1/2$
- D $3/4$
- E 1

2.3 A count-density trapezoid has area $3av_0$, $N=60$ and $v_0=4$ m/s. Its height a is _____. Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: A continuous speed distribution assigns nonzero probability to exactly one specified speed. [2]

3 Exam-style questions

3.1 Two ideal gases at the same temperature have particle masses m and $4m$. Compare their most probable, mean and rms speeds. Compare their mean translational kinetic energies and explain why zero mean velocity does not mean zero energy. [4]

Solutions

2.1 B Velocity directions cancel, while speed is nonnegative and its equilibrium mean is positive.

Compare the rejected options: Mean velocity and mean speed are both zero; The most probable speed is zero; All particles have the rms speed; Speed density has no units. Use the stated model and conditions.

2.2 C Excited total weight $3e^{(-\ln 3)}=1$ equals ground weight 1; $Z=2$.

Compare the rejected options: $1/4$; $1/3$; $3/4$; 1. Use the stated model and conditions.

2.3 5. $a=N/(3v_0)=60/12=5$ particles per unit speed.

2.4 False. Only an interval has nonzero probability; a single point has zero measure.

3.1 All three characteristic speeds scale as $\sqrt{T/m}$, so heavier/light speed ratio is $1/2$. Mean translational energy is $3k_B T/2$ for either gas in the classical regime. Random velocity directions cancel in $\langle \mathbf{v} \rangle$, but $\langle v^2 \rangle$ is positive. Award speed ratios, regime, energy equality and vector distinction.