20.1 Speed distributions and statistical ensembles

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS probability density • partition function

Objective

Normalise continuous speed densities and interpret their units; Distinguish vector means, speed means and most probable speeds; Use Boltzmann weights with degeneracy and a partition function . Original public practice; not an ETS paper.

1 Key knowledge and worked examples

A speed probability density f has ∫f(v)dv=1 and inverse-speed units. A count density P has area N and f=P/N. A trapezoid rising over v0, flat over 2v0 and falling over v0 has area 3av0 for height a. Normalise by area, not height.

Without bulk flow an isotropic gas has zero mean velocity but positive mean speed. Maxwell speed density is proportional to $v^2\exp[-mv^2/(2kBT)]$. Mode, mean and rms speeds are sqrt(2kBT/m), sqrt(8kBT/(πm)) and sqrt(3kBT/m). The v^2 factor makes speed density zero at v=0.

For a continuous distribution, an exact single speed has probability zero; intervals have nonzero probabilities. This does not exclude arbitrarily slow particles or a detector's finite-width zero bin. Characteristic speeds scale as sqrt(T/m), not T.

Canonical state weights are exp(−E/(kBT)); divide by their sum Z. Level probability includes degeneracy g: g exp(−E/(kBT))/Z. Ground degeneracy 1 and excited degeneracy 2 at gap kBTln2 give equal level weights. Classical Maxwell–Boltzmann differs from quantum Bose/Fermi occupations.

■ Worked method

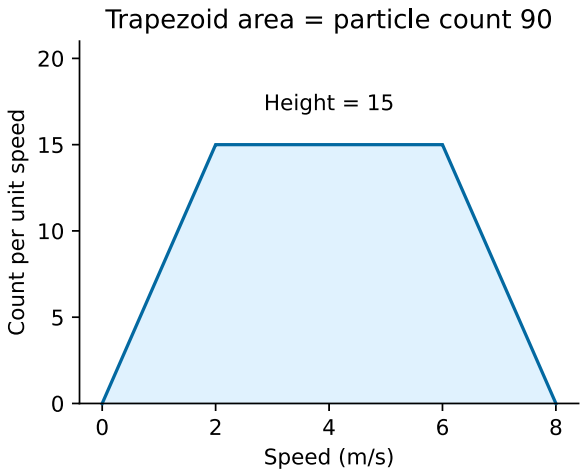
A canonical ensemble has ground degeneracy 1 and excited degeneracy g at gap Delta.

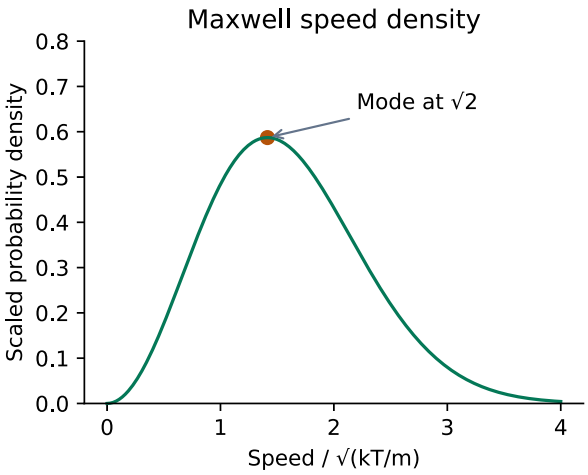
$$Z = 1 + ge^{-\Delta/(k_BT)}, \qquad P_e = \frac{ge^{-\Delta/(k_BT)}}{Z}.$$

With g = 4 and $\Delta = k_BT\ln 2$,

$$P_e = \frac{4e^{-\ln 2}}{1 + 4e^{-\ln 2}} = \frac{2}{3}.$$

This is probability for the entire excited level. Each of its four states has probability 1/6.





2 Practice

2.1 Choose one answer. For an isotropic equilibrium ideal gas without bulk flow, which is correct? [1]

- **A** Mean velocity and mean speed are both zero
- **B** Mean velocity is zero but mean speed is positive
- **C** The most probable speed is zero
- **D** All particles have the rms speed
- **E** Speed density has no units

2.2 Choose one answer. Ground degeneracy 1, excited degeneracy 3, gap kBT ln3. The excited-level probability is: [1]

- **A** 1/4
- **B** 1/3
- **C** 1/2
- **D** 3/4
- **E** 1

2.3 A count-density trapezoid has area 3av0, N=60 and v0=4 m/s. Its height a is _____. Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: A continuous speed distribution assigns nonzero probability to exactly one specified speed. [2]

3 Exam-style questions

3.1 Two ideal gases at the same temperature have particle masses m and 4m. Compare their most probable, mean and rms speeds. Compare their mean translational kinetic energies and explain why zero mean velocity does not mean zero energy. [4]

Solutions

2.1 B Velocity directions cancel, while speed is nonnegative and its equilibrium mean is positive.

Compare the rejected options: Mean velocity and mean speed are both zero; The most probable speed is zero; All particles have the rms speed; Speed density has no units. Use the stated model and conditions.

2.2 C Excited total weight 3e^(−ln3)=1 equals ground weight 1; Z=2.

Compare the rejected options: 1/4; 1/3; 3/4; 1. Use the stated model and conditions.

2.3 5. a=N/(3v0)=60/12=5 particles per unit speed.

2.4 False. Only an interval has nonzero probability; a single point has zero measure.

3.1 All three characteristic speeds scale as $\sqrt{T/m}$, so heavier/light speed ratio is 1/2. Mean translational energy is $3k_BT/2$ for either gas in the classical regime. Random velocity directions cancel in $\langle \mathbf{v} \rangle$, but $\langle v^2 \rangle$ is positive. Award speed ratios, regime, energy equality and vector distinction.