

18.1 Fourier symmetry and wave superposition

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS orthogonality • Fourier coefficient

Objective

Use orthogonality to identify Fourier coefficients; Exploit even and odd parity without discarding the constant term; Relate component amplitudes to interference and physical waveforms. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For a real 2π -periodic function, $f = a_0/2 + \sum [a_n \cos(nx) + b_n \sin(nx)]$. Coefficients are integrals of f times the relevant basis function over $[-\pi, \pi]$, divided by π . Orthogonality separates components. The mean is $a_0/2$, not a_0 .

Even f about the chosen origin has all $b_n = 0$ because $f \sin(nx)$ is odd. Odd f has zero mean and all $a_n = 0$. Shifting the origin can mix sine/cosine coefficients without changing harmonic frequencies. An even nonnegative triangular wave can have cosine terms and a nonzero mean.

For periodic $|x|$ on $[-\pi, \pi]$, integration by parts gives mean $\pi/2$ and $a_n = 2[(-1)^n - 1]/(\pi n^2)$. Thus only odd cosine coefficients survive, at $-4/(\pi n^2)$. Its continuous cusp gives $1/n^2$ decay; discontinuous waveforms need a different convergence analysis.

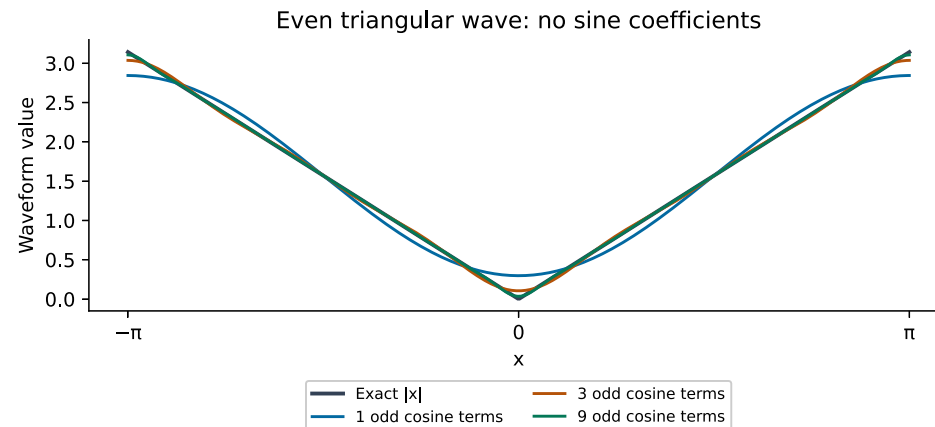
Equal coherent monochromatic fields at phase difference φ give intensity $2I_0(1 + \cos\varphi)$: $4I_0$ in phase and zero at π . Incoherent averaging gives $2I_0$. Fourier sums add fields with their phases; a waveform's instantaneous value is not a sum of individual intensities.

■ Worked method

For a real periodic function, use $f = a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$. Even symmetry makes all sine coefficients zero because the integrand is odd. For $f(x) = |x|$ on $[-\pi, \pi]$,

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) dx = \frac{2[(-1)^n - 1]}{\pi n^2}.$$

Integration by parts supplies the second expression. The mean is $a_0/2 = \pi/2$; only odd cosine harmonics survive. A cusp does not justify dropping the constant term.



2 Practice

2.1 Choose one answer. A real even 2π -periodic waveform about $x=0$ necessarily has: [1]

- A All $a_n = 0$
- B All $b_n = 0$
- C Zero mean
- D Only its fundamental
- E Equal a_n and b_n

2.2 Choose one answer. For the periodic extension of $|x|$, a_2 is: [1]

- A $-4/\pi$
- B $-1/\pi$
- C 0
- D $\pi/2$
- E 2

2.3 If $a_0 = 6$, the waveform mean is _____. Show your relation and units. [2]

[2]

[4]

Solutions

2.1 B Even times sine is odd, so each symmetric sine-coefficient integral is zero.

Compare the rejected options: All $a_n=0$; Zero mean; Only its fundamental; Equal a_n and b_n . Use the stated model and conditions.

2.2 C $a_n=2[(-1)^{n-1}]/(\pi n^2)$; even n gives zero.

Compare the rejected options: $-4/\pi$; $-1/\pi$; $\pi/2$; 2. Use the stated model and conditions.

2.3 3. The constant term is $a_0/2$.

2.4 True. The angle-addition identities mix coefficients at the same frequency.

3.1 Substitute the shift in $f \approx \pi/2 - (4/\pi) \cos x$: $g \approx \pi/2 - (4/\pi) \sin x$. Its mean remains $\pi/2$. Evenness depends on the chosen origin; g is not even about zero. Frequencies stay unchanged. Award substitution, harmonic/sign, mean and origin/frequency interpretation.