

Exercise walk-through

37.1

Weak perturbations and degenerate subspaces

GRE · GRE Subject Test · GRE Physics

37.1 Weak perturbations and degenerate subspaces

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS first-order energy shift

- degenerate perturbation theory

Objective

Calculate nondegenerate first-order energy shifts as normalised expectation values; Use symmetry and matrix elements to assess weak-coupling limits; Diagonalise the perturbation within an exactly degenerate subspace . Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For $H=H_0+\lambda W$ and normalised nondegenerate $|n\rangle$, first-order shift is $\lambda\langle n|W|n\rangle$. A weak γx potential in $0<x<L$ gives $\gamma L/2$, not the potential at a chosen point.

Centred odd perturbations have zero diagonal expectation for a parity state. A centred quadratic shift is $\eta L^2[1/12 - 1/(2\pi^2 n^2)]$. Zero first order does not imply an unchanged exact state.

Weak mixing needs $|\lambda W_{mn}|$ small compared with $|E_n^0 - E_m^0|$. Second-order terms use squared couplings divided by signed gaps; the ground-state denominators are negative.

For exact degeneracy, diagonalise the perturbation matrix within the subspace. $\varepsilon[[2,1],[1,2]]$ has shifts ε and 3ε , from antisymmetric/symmetric combinations. Diagonal entries alone are insufficient.

■ **Worked method**

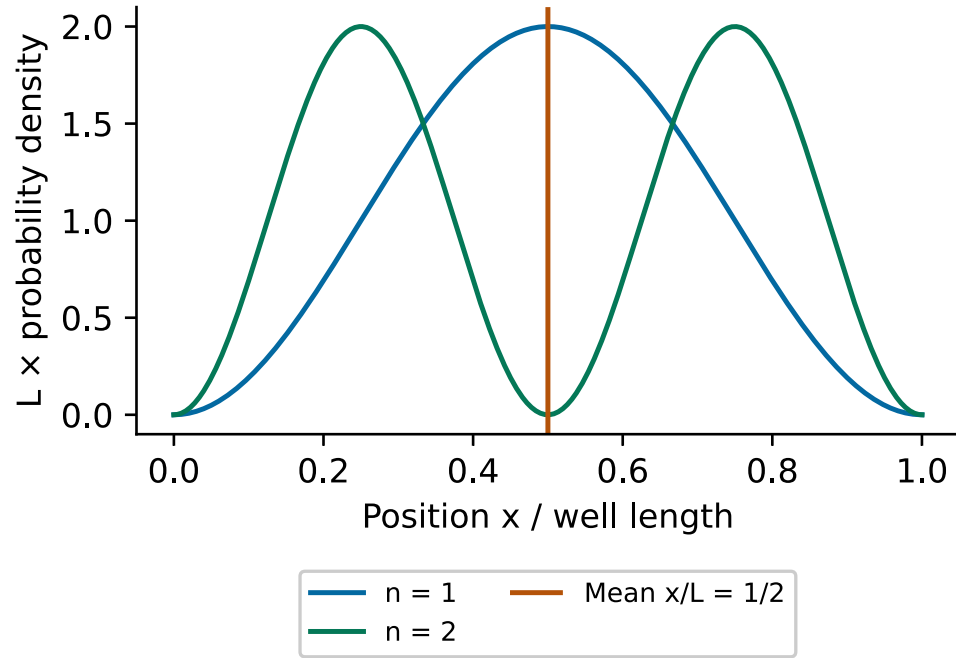
A degenerate subspace requires matrix diagonalisation before first-order shifts. Suppose

the perturbation restricted to two states is $\epsilon \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

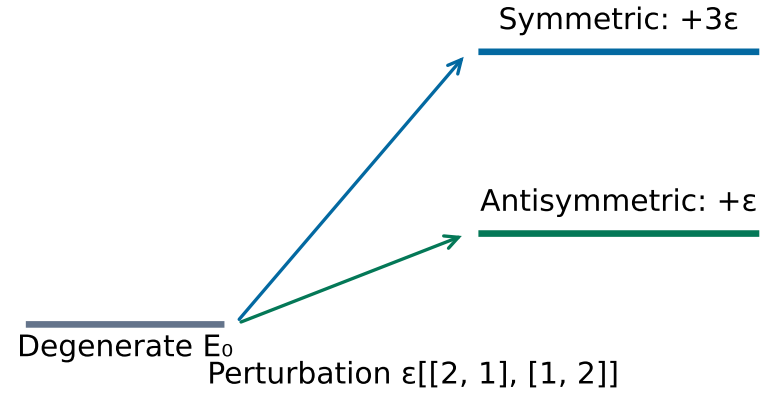
$$\det(W - wI) = (2\epsilon - w)^2 - \epsilon^2 = 0.$$

Shifts are ϵ and 3ϵ . Normalised states are $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$. Reading only the diagonal entries would miss the splitting. Couplings to other levels must remain weak relative to their energy gaps.

Different densities; same linear mean



Diagonalise the complete degenerate block



Exercise 2.1

2.1 Choose one answer. In $0 < x < L$, a weak perturbation $\gamma(x - L/2)$ acts on an unperturbed infinite-well eigenstate. Its first-order energy shift is: [1]

- **A** γL
- **B** $\gamma L/2$
- **C** 0
- **D** $-\gamma L/2$
- **E** Undefined for every n

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- **D** $-\gamma L/2$
- **E** Undefined for every n

2.1 C The density is symmetric about $L/2$ while the perturbation is odd, so the integral cancels.

Compare the rejected options: γL ; $\gamma L/2$; $-\gamma L/2$; Undefined for every n . Use the stated model and conditions.

Exercises 2.2·2.3

2.2 Choose one answer. A two-state degenerate block has perturbation $\varepsilon[[0,1],[1,0]]$, $\varepsilon > 0$. First-order shifts are: [1]

- **A** 0 and 0
- **B** ε and ε
- **C** $-\varepsilon$ and $+\varepsilon$
- **D** 0 and 2ε
- **E** -2ε and 2ε

2.3 For a 4 nm well and weak γx with $\gamma = 0.02$ eV/nm, first-order shift is _____ eV. Show your relation and units. [2]

Exercises 2.2-2.3

2.2 Choose one answer. A two-state degenerate block has perturbation $\varepsilon[[0,1],[1,0]]$, $\varepsilon > 0$. First-order shifts are: [1]

- **A** 0 and 0
- **B** ε and ε
- **C** $-\varepsilon$ and $+\varepsilon$
- **D** 0 and 2ε
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2.2 C The matrix eigenvalues are $\pm\varepsilon$. Its vanishing diagonal does not mean vanishing splitting.

Compare the rejected options: 0 and 0; ε and ε ; 0 and 2ε ; -2ε and 2ε . Use the stated model and conditions.

2.3 For a 4 nm well and weak γx with $\gamma = 0.02$ eV/nm, first-order shift is _____ eV. Show your relation and units. [2]

2.3 0.04. $\gamma \langle x \rangle = 0.02 \cdot 4/2 = 0.04$ eV.

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: A zero diagonal first-order expectation guarantees all higher-order energy shifts are zero. [2]

3.1 A two-state Hamiltonian is $\begin{pmatrix} 0 & \epsilon \\ \epsilon & \Delta \end{pmatrix}$, with Δ positive and $|\epsilon| \ll \Delta$. Find the exact eigenvalues and leading ground-energy shift. Explain why a zero diagonal perturbation is insufficient to predict zero shift. [5]

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: A zero diagonal first-order expectation guarantees all higher-order energy shifts are zero. [2]

2.4 False. Off-diagonal couplings can contribute at second order even when the diagonal expectation cancels.

3.1 A two-state Hamiltonian is $\begin{pmatrix} 0 & \epsilon \\ \epsilon & \Delta \end{pmatrix}$, with Δ positive and $|\epsilon| \ll \Delta$. Find the exact eigenvalues and leading ground-energy shift. Explain why a zero diagonal perturbation is insufficient to predict zero shift. [5]

3.1 Characteristic equation is $E^2 - \Delta E - \epsilon^2 = 0$. $E_{\pm} = (\Delta \pm \sqrt{\Delta^2 + 4\epsilon^2})/2$. Expand the lower root: $E_- \approx -\epsilon^2/\Delta$. The first-order shift is zero, but off-diagonal mixing produces a negative second-order shift. Award equation, roots, expansion, sign and mixing/validity. At $\Delta = 0$ use degenerate theory instead.