# T.12 · Metric completeness and closure from a basis

Original classroom teaching; not an official form or score predictor

# Metric completeness and closure from a basis

- Check the metric axioms.
- Decide whether a space is a **complete metric space 完备度量空间**.
- Use pullback metrics and isometries.
- Find closures and interiors from a basis or a **neighbourhood basis 邻域基**.

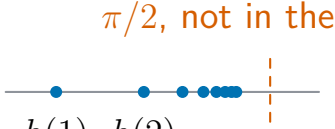
# Metrics

- $d(x, y) \geq 0$, with $d(x, y) = 0$ exactly when $x = y$.
- Symmetry: $d(x, y) = d(y, x)$.
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
- $|x - y|^2$ is not a metric on $\mathbb{R}$: $d(0, 2) = 4 > d(0, 1) + d(1, 2) = 2$.

# Completeness

- A space is complete when every Cauchy sequence converges to a point of the space.
- $\mathbb{R}$ is complete; $\mathbb{Q}$ and $(0, 1)$ are not. A closed subset of a complete space is complete.
- If $f$ is injective, $d(x, y) = |f(x) - f(y)|$ is a metric, and $f$ is an isometry onto its image. The space is complete exactly when the image is complete.
- Completeness depends on the metric. $\mathbb{R}$ with $|\arctan x - \arctan y|$ has the usual open sets but is not complete.

# Figure: Completeness




$d(m,n) = |\arctan m - \arctan n|$: the integers become Cauchy but converge to nothing, because the image is not complete

- A pullback metric is complete only if the image is.
- Match each part of the diagram to the text.

# Bases and closures

- A point $x$ is in the closure of $A$ exactly when every basic open set containing $x$ meets $A$.
- A point is in the interior of $A$ when some basic open set containing it lies inside $A$.
- Changing the basis can change the closure, even for the same set.

# Worked example: two pullback metrics

- Both are metrics: $x^3$ and $\arctan x$ are injective, and absolute value gives symmetry and the triangle inequality.
- $x^3$ maps $\mathbb{R}$ onto $\mathbb{R}$, which is complete, so $d_1$ is complete.
- $\arctan$ maps onto $(-\pi/2, \pi/2)$. The sequence $x_n = n$ is $d_2$-Cauchy, since $\arctan n \to \pi/2$, but no real number has image $\pi/2$. So $d_2$ is not complete.

# Worked example: a different basis

- The point 0: every basic set $[0, \varepsilon)$ meets $A$, so 0 is in the closure.
- The point 1: the basic set $[1, 1.5)$ does not meet $A$, so 1 is not in the closure.
- The closure is $[0, 1)$, while in the usual topology it is $[0, 1]$.

# Common errors

- Assuming a bounded metric space is complete.
- Treating completeness as a property of the open sets alone.
- Forgetting to check the triangle inequality.

# Common errors (2)

- Using the usual closure when a different basis is given.
- Thinking a closed set in an incomplete space must be complete.

# Key terms

- **complete metric space**: A metric space in which every Cauchy sequence converges to a point of the space.
- **neighbourhood basis**: A collection of open sets around a point that is enough to test closeness to that point.

# Check yourself

- Is $\mathbb{Q}$ with the usual distance complete?
- Is $[0, \infty)$ with the usual distance complete?
- In the lower-limit topology, is $[0, 1)$ open?