C.12 · Related rates and removable quotient limits

Original classroom teaching; not an official form or score predictor

Related rates and removable quotient limits

- Set up and solve related-rate problems with correct signs and units.
- Find a **removable limit 可去极限** by factoring or rationalising.
- Use standard limits and the squeeze theorem.

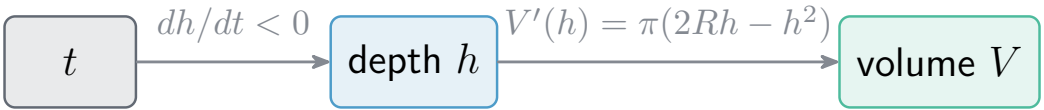
Related rates

- Draw the situation and name every changing quantity.
- Write an equation that holds at every moment.
- Differentiate it with respect to time, using the chain rule.

Related rates (2)

- Only then substitute the values at the moment asked about. Substituting first turns variables into constants.
- A decreasing quantity has a negative rate; report a magnitude only if the question asks for one.

Figure: Related rates



$$\frac{dV}{dt} = V'(h)\,\frac{dh}{dt} \qquad \text{(negative while draining)}$$

- Related rates through the chain rule.
- Match each part of the diagram to the text.

Removable limits

- Factor and cancel: $\dfrac{x^2-9}{x-3} = x+3$ for $x \neq 3$, so the limit is 6.
- Rationalise: $\dfrac{\sqrt{x+4}-2}{x} = \dfrac{1}{\sqrt{x+4}+2} \to \dfrac{1}{4}$.
- Defining $f(a)$ as the limit fills the gap and makes the function continuous there.

Standard limits

- $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ (with x in radians), so $\lim_{x\to 0} \dfrac{\sin 3x}{x} = 3$.
- $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$ and $\lim_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$.
- Squeeze: if $g \le f \le h$ and g, h have the same limit, so does f. $x^2 \sin(1/x) \to 0$ since $|x^2 \sin(1/x)| \le x^2$.

Worked example: a spherical tank

- A slice at height s has radius squared $2Rs - s^2$, so $V(h) = \pi\left(Rh^2 - \dfrac{h^3}{3}\right)$.
- $\dfrac{dV}{dt} = \pi(2Rh - h^2)\dfrac{dh}{dt} = \pi(4-1)(-0.1) = -0.3\pi$ m^3/min.
- Water leaves at $0.3\pi \approx 0.94$ m^3/min; the minus sign records the loss.

Worked example: a sliding ladder

- $x^2 + y^2 = 25$, so $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$.
- At $x = 3$, $y = 4$: $3 \times 0.5 + 4\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -0.375$ m/s.
- The top falls at 0.375 m/s; it speeds up as the ladder gets lower.

Common errors

- Substituting values before differentiating.
- Losing the sign of a decreasing quantity.
- Mixing units, such as cm and m, in one rate.

Common errors (2)

- Using $\sin x/x \to 1$ with x in degrees.
- Concluding a limit does not exist because the function is undefined at the point.

Key terms

- **related rate**: A rate of change found by differentiating an equation that links changing quantities.
- **removable limit**: A finite limit at a point where the function is undefined, which can fill the gap continuously.

Check yourself

- A circle's radius grows at 2 cm/s. How fast is the area growing when $r = 5$ cm?
- What is $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$?
- What is $\lim_{x \to 0} \dfrac{\tan x}{x}$?